SECOND HALF TERM ROUND UP

MATHS SOCIETY SPEAKERS

Contents

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Part 1. Saga 2- Generalising Connections Between the Golden ratio and Fibonacci Numbers

1. Continued Fractions

1.1. Introduction. In this talk we looked at the idea of continued fractions. We begin with a simple example

$$\frac{43}{19} = 2 + \frac{5}{19}$$
$$= 2 + \frac{1}{\frac{19}{5}} = 2 + \frac{1}{3 + \frac{4}{5}}$$
$$= 2 + \frac{1}{\frac{19}{5}} = 2 + \frac{1}{3 + \frac{1}{\frac{5}{4}}}$$
$$= 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}}$$

Thus, we see that even a simple fraction admits an interesting continued fraction expansion. But the theory of continued fractions, as we shall see provides us with some deep connections to number theory...

1.2. Application to Quadratics. Now consider a quadratic equation, for example $x^2 - 5x - 1 = 0$. Normally, to solve this we would use the quadratic formula, but here we shall use the language of continued fractions:

$$x^2 = 5x + 1$$
$$x = 5 + \frac{1}{x}$$
$$x = 5 + \frac{1}{5 + \frac{1}{x}}$$
$$x = 5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\cdots}}}}$$

And so we've expressed the solution to this quadratic as an infinite continued fraction! But now let us dive deeper and look at a connection to the golden ratio. Recall that the golden ratio is the solution to the equation

$$\varphi^2 = \varphi + 1.$$

And so we shall rearrange it:

$$\varphi = 1 + \frac{1}{\varphi}$$
$$\varphi = 1 + \frac{1}{1 + \frac{1}{\varphi}}$$
$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\cdots}}}$$

Or in other notation, $\varphi = [1; 1, 1, 1, \ldots]$. It turns out that this continued fraction has a shocking connection to the Fibonacci numbers...

1.3. **Two Shocking Connections.**

Definition 1.1. Remember that the Fibonacci numbers F_n are defined by the relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. The first few numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

We observe that the convergents of the golden ratio is precisely the ratio of the Fibonacci numbers. Let us the first few incidents:

$$1, 1 + \frac{1}{1}, 1 + \frac{1}{1 + \frac{1}{1+1}}, 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}, \ldots$$

which becomes:

$$\frac{1}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$$

As we can see, as we go through the convergents of the infinite continued fraction of φ, we keep getting ratios of two Fibonacci numbers. From this we can also infer the cool fact that as we take this process to infinity, we have:

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi$$

We now look at a second shocking connection to number theory which we observed- a connection to Pell numbers. First, we have to find the continued fraction expansion for $\sqrt{2} + 1$. This is the solution to

$(x-1)^2 = 2$ and so we obtain:

$$x^2 = 2x + 1$$
$$x = 2 + \frac{1}{x}$$
$$x = 2 + \frac{1}{2 + \frac{1}{x}}$$
$$x = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\cdots}}}$$

Now we look at how this connects to Pell numbers.

Definition 1.2. The Pell numbers are defined by $P_n = 2P_{n-1} + P_{n-2}$ with $P_0 = 0$, $P_1 = 1$. The first few Pell numbers are: $0, 1, 2, 5, 12, 29, 70, \ldots$.

The convergents of $x = \sqrt{2} + 1$ are the following:

$$2, 2 + \frac{1}{2}, 2 + \frac{1}{2 + \frac{1}{2}}, 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \ldots$$

which become

$$\frac{2}{1}, \frac{5}{2}, \frac{12}{5}, \frac{29}{12}, \ldots$$

and again we see how the convergents of a certain special continued fraction are deeply related to a famous sequence in number theory. Fun fact: the number $1 + \sqrt{2}$ is called the silver ratio. Next time we shall look in more depth into the Fibonacci numbers and the golden ratio.

2. On the Fibonacci Numbers and Golden Ratio

2.1. **Introduction.** In the last talk we unveiled a shocking connection between the continued fraction of the golden ratio and the Fibonacci numbers (check it out if you haven't seen the notes from that talk). This week we delve deeper into the connection between the golden ratio and the Fibonacci numbers, and uncover some deep results about the Fibonacci numbers.

2.2. **Deriving the General Formula for F_n.** We will now use an ingenious argument to find a general formula for the nth Fibonacci numbers. First recall that the golden ratio satisfies $\varphi^2 = \varphi + 1$ and so we have:

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2}.$$

We also have that the other solution to the quadratic $\psi := 1 - \varphi = \frac{-1}{\varphi}$ satisfies this too. Thus in general, any sequence that satisfies the recursion $U_n = U_{n-1} + U_{n-2}$ will be of the form:

$$U_n = a\varphi^n + b\psi^n$$

since

$$\begin{aligned} U_n &= a\varphi^n + b\psi^n \\ &= a(\varphi^{n-1} + \varphi^{n-2}) + b(\psi^{n-1} + \psi^{n-2}) \\ &= U_{n-1} + U_{n-2} \end{aligned}$$

as required. Now we apply this to F_n. Since $F_0 = 0$ and $F_1 = 1$, we have that:

$$\begin{aligned} a + b &= 0 \\ a\varphi + b\psi &= 1 \end{aligned}$$

And once the algebra is done, you have: $a = \frac{1}{\sqrt{5}}$, $b = \frac{-1}{\sqrt{5}}$ and so we have:

$$\boxed{F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}} \tag{1}$$

(using $\psi = -\varphi^{-1}$).

2.3. **Developing a Test For Fibonacci Numbers.** Using the equation we derived in the previous section, we can develop a test to see if a random integer x is a Fibonacci number, meaning you don't have to go through all of the Fibonacci numbers to test it! First we must prove a preliminary lemma:

Lemma 2.1. $\varphi^n = \varphi F_n + F_{n-1}$

Proof. We do this by induction which means we prove it for the first case ($n = 1$) and then we assume that it is true for the nth case and show that this implies that the statement is true for the $(n+1)$th case. So let us proceed:
The base case: $\varphi^1 = \varphi(1) + 0$ which is clearly true.
The inductive step: Let us assume that $\varphi^n = \varphi F_n + F_{n-1}$. Then:

$$\begin{aligned} \varphi^{n+1} &= \varphi^2 F_n + \varphi F_{n-1} \\ &= (\varphi + 1)F_n + \varphi F_{n-1} \\ &= \varphi(F_n + F_{n-1}) + F_n \\ &= \varphi F_{n+1} + F_n \end{aligned}$$

as required. $\square$

Now we can go about making this test. Using 1 we have that:

$$\begin{aligned}
&\sqrt{5}F_n = \varphi^n - (-\varphi)^{-n}\\
&\varphi^n\sqrt{5}F_n = \varphi^{2n} - (-1)^n\\
&\varphi^{2n} - \varphi^n\sqrt{5}F_n - (-1)^n = 0\\
&\varphi^n = \frac{\sqrt{5}F_n + \sqrt{5F_n^2 + 4(-1)^n}}{2},
\end{aligned}$$

where the last step comes from the quadratic formula (if look closely, it's a quadratic of φ^n). Now using the lemma we derived before, we have:

$$\begin{aligned}
&\varphi F_n + F_{n-1} = \frac{\sqrt{5}F_n + \sqrt{5F_n^2 + 4(-1)^n}}{2}\\
&= 2\varphi F_n + 2F_{n-1} = \sqrt{5}F_n + \sqrt{5F_n^2 + 4(-1)^n}\\
&\overset{2\varphi=1+\sqrt{5}}{\implies} (1+\sqrt{5})F_n + 2F_{n-1} = \sqrt{5}F_n + \sqrt{5F_n^2 + 4(-1)^n}\\
&\implies F_n + 2F_{n-1} = \sqrt{5F_n^2 + 4(-1)^n}\\
&\implies (F_n + 2F_{n-1})^2 = 5F_n^2 + 4(-1)^n.
\end{aligned}$$

And here we are at the crux of the argument! Since the right hand side is always a perfect square, we can say that x is a Fibonacci number iff $5x^2 - 4$ is a perfect square or $5x^2 + 4$ is a perfect square. Pretty nifty, eh!

3. Pell Numbers, The Silver Ratio and More Generalisations

3.1. **Introduction.** This week we finish the saga on all that we've been looking at in number theory for the last few weeks. And this will hopefully be an exciting end!

3.2. **Pell Numbers and The Silver Ratio.** The silver ratio is defined in a very similar way to the golden ratio. Whilst the golden ratio is defined as the solution to $x^2 - x - 1 = 0$, the silver ratio shall be defined as the positive solution to $x^2 - 2x - 1 = 0$. In other words, the silver ratio is $1 + \sqrt{2}$. We can observe that the silver ratio enjoys a very similar continued fraction expansion to the golden ratio:

$$\begin{aligned} x^2 - 2x - 1 &= 0 \\ x^2 &= 2x + 1 \\ x &= 2 + \frac{1}{x} \\ &= 2 + \frac{1}{2 + \frac{1}{x}} \\ &= 2 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\dots}}} \end{aligned}$$

which looks exactly like the golden ratio, except for that one the 2s were replaced with 1s (see the notes from that legendary talk last week for a refresher). However, here the fun has just begun! Because recall that last time we observed a connection between the golden ratio and the Fibonacci numbers (which were defined with the recurrance relation $F_n = F_{n-1} + F_{n-2}$.) So this time, the silver ratio will have a shocking connection with the very similar Pell numbers which are defined by $P_n = 2P_{n-1} + P_{n-2}$ (with $P_0 = 0$ and $P_1 = 1$). So the first few Pell numbers would be: $0, 1, 2, 5, 12, 29, 70, \dots$. Again there is a shocking connection: the convergents of the silver ratio are the following:

$$2, 2 + \frac{1}{2}, 2 + \frac{1}{2 + \frac{1}{2}}, 2 + \frac{1}{2 + \frac{1}{2+\frac{1}{2}}}, \dots$$

which become

$$\frac{2}{1}, \frac{5}{2}, \frac{12}{5}, \frac{29}{12}, \dots$$

Furthermore, just as we did with the Fibonacci numbers, we can now use the silver ratio to get a formula for the nth Pell number. Because we can simply multiply $x^2 = 2x + 1$ by x^{n-2} on both sides, we have that $x^n = 2x^{n-1} + x^{n-2}$ and so since the solutions are $x = 1 \pm \sqrt{2}$, we know the formula for the nth Pell number will be of the form

$$P_n = a(1 + \sqrt{2})^n + b(1 - \sqrt{2})^n$$

since

$$\begin{aligned} P_n &= a(1 + \sqrt{2})^n + b(1 - \sqrt{2})^n \\ &= 2a(1 + \sqrt{2})^{n-1} + 2b(1 - \sqrt{2})^{n-2} + a(1 + \sqrt{2})^{n-1} + b(1 - \sqrt{2})^{n-2} \\ &= 2P_{n-1} + P_{n-2} \end{aligned}$$

as required. So now we must find a and b. We have set $P_0 = 0$ and $P_1 = 1$, so we just plug these two values in and get simultaneous equations and then we should be fine and dandy. More specifically, we get:

$$P_0 = a(1+\sqrt{2})^0 + b(1-\sqrt{2})^0 = a+b = 0$$
$$P_1 = a(1+\sqrt{2}) + b(1-\sqrt{2}) = 1.$$

After you do the algebra, you get that $a = \frac{1}{2\sqrt{2}}$ and $b = \frac{-1}{2\sqrt{2}}$. So we obtain a glorious formula for the nth Pell number being:

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

So again we see that there is a connection between these special ratios and some very nice sequences in number theory. Thus, as any mathematician would seek to do, we generalised it and saw the full story!

3.3. **The Maths Society Ratio.** Yes, I know these are called the metallic means, but we generalised it ourselves and the results agreed with what mathematicians had inevitably already done so I called it the maths society ratio: sue me! Anyway, remember that the quadratic equations defining the golden and silver ratios ,respectively, were $x^2 - x - 1 = 0$ and $x^2 - 2x - 1 = 0$. So naturally we then looked at $x^2 - nx - 1 = 0$, and called this the maths society ratio (which we denoted δ because it was Mithush's favourite Greek letter). Anyway, let us observe that $\delta(= \frac{n+\sqrt{n^2+4}}{2})$ enjoys a very similar continued fraction expansion to the golden and silver ratios:

$$\begin{aligned} x^2 - nx - 1 &= 0 \\ x^2 &= nx + 1 \\ x &= n + \frac{1}{x} \\ &= n + \frac{1}{n + \frac{1}{x}} \\ &= n + \frac{1}{n + \frac{1}{n + \frac{1}{n+\dots}}} \end{aligned}$$

We now see how this links to the sequence $M_i = nM_{i-1} + M_{i-2}$, with $M_0 = 0$ and $M_1 = 1$. For example, when $n = 3$ the first few terms would be: $0, 1, 3, 10, 33, 109, \dots$. And the convergents of the continued fraction would be:

$$3, 3 + \frac{1}{3}, 3 + \frac{1}{3 + \frac{1}{3}}, 3 + \frac{1}{3 + \frac{1}{3+\frac{1}{3}}}, \dots$$

which become

$$\frac{3}{1}, \frac{10}{3}, \frac{33}{10}, \frac{109}{33}, \ldots$$

And in general the convergents of the maths society ratio for any n are

$$n, n + \frac{1}{n}, n + \frac{1}{n + \frac{1}{n}}, n + \frac{1}{n + \frac{1}{n+\frac{1}{n}}}, \ldots$$

which become

$$\frac{n}{1}, \frac{n^2+1}{n}, \frac{n(n^2+1)+n}{n^2+1}, \frac{n(n(n^2+1)+n)+n^2+1}{n(n^2+1)+n}, \ldots$$

which is precisely just the first few terms of our sequence in a fraction. The last thing we did was find a general formula for M_i. Using the same reasoning again it will be of the form $M_i = a\delta^i + b\bar{\delta}^i$, where $\bar{\delta}$ is $\frac{n-\sqrt{n^2+4}}{2}$. And again, doing the algebra on the simultaneous equations

$$a + b = 0$$
$$a\delta + b\bar{\delta} = 1$$

yields $a = \frac{1}{\sqrt{n^2+4}}$ and $b = \frac{-1}{\sqrt{n^2+4}}$ and so our general formula is:

$$\boxed{M_i = \frac{\delta^i - \bar{\delta}^i}{\sqrt{n^2+4}}}$$

(fun exercise: check that this agrees with the formulae for the nth Fibonacci and nth Pell number.)

3.4. **Further Reading.** Thank you to Saik for pointing out this link which had some more amazing results on the metallic means- we barley scratched the surface in this saga!

Part 2. **Saga 3- Transcendental Numbers**

4. Transcendental Numbers Exist- A Look At Liouville Numbers

4.1. **Introduction.** Last time it was revealed that our next saga would be a look into the world of transcendental numbers, which are numbers that are not a root of any equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $a_i \in \mathbb{Z}$. In this talk we establish the fact that these numbers exist by looking at the first example that was discovered in 1844- the Liouville constant, which is an example of a class of transcendental numbers known as Liouville numbers

4.2. **Liouville Numbers.** The goal of this talk is to prove that the number $L = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is irrational. We do this in the following way:

(1) Prove that all Liouville numbers are transcendental
(2) Show that L is a Liouville number.

Sadly, the definition of a Liouville number is pretty technical.

Definition 4.1. A Liouville number α is a number such that for all $n \in \mathbb{N}$, there exists a rational number $\frac{a}{b}$ (with $b > 1$) such that:

$$0 < \left|\alpha - \frac{a}{b}\right| < \frac{1}{b^n}.$$

Proposition 4.1. If α is a Liouville number, then α is irrational.

Proof. Let us assume that α is a rational Liouville number $\frac{p}{q}$. Then for some rational number $\frac{a}{b} \neq \frac{p}{q}$, we have:

$$0 < \left|\alpha - \frac{a}{b}\right| = \left|\frac{p}{q} - \frac{a}{b}\right| = \left|\frac{pb - aq}{qb}\right|$$

Now we pick a natural number n such that $2^{n-1} > q$. Then we have:

$$\left|\frac{pb - aq}{qb}\right| > \frac{1}{2^{n-1}b} \geq \frac{1}{b^n}$$

where the last inequality sign comes from the fact that $b > 1$. Thus we have shown that for any $\frac{a}{b}$ we try to choose, there will be an $n \in \mathbb{N}$ such that $\left|\alpha - \frac{a}{b}\right| > \frac{1}{b^n}$ which contradicts the assumption that α was a Liouville number. Thus, all Liouville numbers are irrational. □

Now that we have established that all Liouville numbers are irrational, we can now move forward and try to prove that they are also transcendental too.

Theorem 4.2. *Liouville numbers are transcendental.*

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_0$ and assume by contradiction that $f(\alpha) = 0$. We now define a few things: $M = \max_{[\alpha-1,\alpha+1]} |f'(x)|$, $A < \{1, \frac{1}{M}, |\alpha - \alpha_1|, \ldots |\alpha - \alpha_m|\}$ where $f(\alpha_i) = 0, \forall i \leq m$. Then we pick some $r \in \mathbb{N}$ such that $2^r \geq \frac{1}{A}$. Since α is a Liouville number, we have some $\frac{a}{b} \in \mathbb{Q}$ such that:

$$0 < \left|\alpha - \frac{a}{b}\right| < \frac{1}{b^{n+r}} \leq \frac{1}{2^r b^n} \leq \frac{A}{b^n} < A. \tag{2}$$

Now, because $\left|\alpha - \frac{a}{b}\right| < A$, we have that:

(1) $\frac{a}{b} \in [\alpha - 1, \alpha + 1]$
(2) $f\left(\frac{a}{b}\right) \neq 0$.

Thus we can use the mean value theorem to say that there exists some $x_0 \in \left(\alpha, \frac{a}{b}\right)$ such that

$$f'(x_0) = \frac{f(\alpha) - f\left(\frac{a}{b}\right)}{\alpha - \frac{a}{b}} = \frac{-f\left(\frac{a}{b}\right)}{\alpha - \frac{a}{b}}$$

So we can say that:

$$|f'(x_0)| = \frac{\left|f\left(\frac{a}{b}\right)\right|}{\left|\alpha - \frac{a}{b}\right|} \implies \left|\alpha - \frac{a}{b}\right| = \frac{\left|f\left(\frac{a}{b}\right)\right|}{|f'(x_0)|} \geq \frac{\left|f\left(\frac{a}{b}\right)\right|}{M} \tag{3}$$

Now note that we have:

$$\left|f\left(\frac{a}{b}\right)\right| = \left|a_0 + \frac{a_1 a}{b} + \cdots + \frac{a_n a^n}{b^n}\right| = \frac{1}{b^n}|b^n a_0 + a_1 a b^{n-1} + \cdots + a_n a^n| \geq \frac{1}{b^n}.$$

So finally we are done because now we plug this into 3 to obtain:

$$\frac{\left|f\left(\frac{a}{b}\right)\right|}{M} \geq \frac{1}{b^n M} > \frac{A}{b^n} > \left|\alpha - \frac{a}{b}\right|. \tag{4}$$

But 2 and 4 imply that $\left|\alpha - \frac{a}{b}\right| > \left|\alpha - \frac{a}{b}\right|$ which is a clear contradiction so there could not have existed some f such that $f(\alpha) = 0$, so α is transcendental as required! □

4.3. **Liouville Constant.** We now show that $L = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is a Liouville number.

Proposition 4.3. *L is a Liouville number.*

Proof. Let us write L as $\sum_{n=1}^{m} \frac{1}{10^{n!}} + \sum_{n=m+1}^{\infty} \frac{1}{10^{n!}}$. Then this first part can be collapsed into a single fraction of the form $\frac{a}{10^{n!}}$. We shall pick our b in the fraction $\frac{a}{b}$ to be $10^{n!}$. Then:

$$0 < \left|L - \frac{a}{b}\right| = \left|\sum_{n=m+1}^{\infty} \frac{1}{10^{n!}}\right| < \frac{1}{b^n}\left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) = \frac{1}{b^n}$$

where the last inequality comes from simply comparing the denominators of the fractions on the sums and observing that they are bigger in the sum on the left and so we have shown that L satisfies the conditions being a Liouville number and so L is transcendental! □

5. e AND π ARE TRANSCENDENTAL

I was lacking that week, so we just watched a YouTube video. The notes are in this link: here

Part 3. Guest Talks

6. The Riemann Zeta Function, or, How to Win £1000000- Wren Shakespeare

6.1. On the Real Line. We begin by considering two related problems from the earlier days of maths. The first is the harmonic series:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\dots$$

This series was proved to equal ∞ by Nicole Oresme in 1350. The second is the Basel problem:

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\dots$$

This series has a finite value that is now quite well-known:

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\dots=\frac{\pi^2}{6}$$

This was proven, if very unrigorously, by Leonhard Euler, who is also credited with the first use of the modern-day Riemann Zeta function, which is a generalisation from both of these ideas:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

This infinite series only produces a finite value in a traditional sense for $s > 1$, as can be seen by comparison with the harmonic series - $s < 1$ means all the terms will be bigger than the harmonic series, so its value will be greater than the harmonic series, which is infinite. Euler also proved, again slightly unrigorously in some cases, two interesting properties of the zeta function:

$$\zeta(2n)=k\pi^{2n}, k\in\mathbb{Q}, n\in\mathbb{N}$$
$$\zeta(s)=\frac{1}{1-2^{-s}}\times\frac{1}{1-3^{-s}}\times\frac{1}{1-5^{-s}}\times\frac{1}{1-7^{-s}}\dots, s>1$$

By contrast, very little is known about the zeta function on odd numbers - the most we know is that $\zeta(3)$ is irrational.

6.2. Complex Numbers. The zeta function was most famously considered as a complex function by Bernhard Riemann, which is why it bears his name today. It can be extended to a complex function for $\mathrm{Re}(s) > 1$ very easily. However, what if we want to define it for other numbers?

$$\text{Taking } \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} :$$

$$\text{We can show that } \zeta(s) = \frac{\eta(s)}{1-2^{1-s}}, \mathrm{Re}(s) > 1$$

However, η also converges for $0 \leq \mathrm{Re}(s) \leq 1$, giving us a handy way to extend the zeta function to this region, which is exactly the region we want to study.
The Riemann Hypothesis asserts that if $\zeta(s) = 0$, then $\mathrm{Re}(s) = \frac{1}{2}$. Initially discarded as a mere curiousity by Riemann himself, it became increasingly apparent that RH implies a number of other things in various fields of maths. Notably, it implies a fairly regular distribution of prime numbers.

So, how far have we gotten with proving it? We can prove there are infinitely many zeroes of the form $\mathrm{Re}(s) = \frac{1}{2}$. We also know, if there are zeroes with other real parts, they cannont be too close to 0 or 1 (how close depends on their exact location). The strongest result that we currently have is that 40% of the zeroes are of the form $\mathrm{Re}(s) = \frac{1}{2}$. Interestingly, the monetary prize is not for a resolution of the Riemann Hypothesis one way or another, it's for a proof (so if it was false, whoever showed that would not be able to collect the prize).

7. Introduction to Taylor Series- Prasan Patel

7.1. **Introduction.** In this talk we were introduced to the concept of a Taylor series, which was named after English mathematician Brook Taylor who was born in 1685 here in Edmonton! So we were blessed to see the magic of Taylor series delivered in its hometown. The idea behind Taylor series is to convert functions that are not polynomials into "infinite polynomials", so that they are easier to compute; a calculator doesn't actually know what e^2 is- it is just plugging in values into the Taylor series to approximate it.

7.2. **Example: e^x.** Our goal is to write e^x in the form $p(x) = c_0 + c_1x + c_2x^2 + \dots$. To do this we shall look at the derivatives. It is well known that $\frac{d}{dx}e^x = e^x$ and so when $x = 0$, the derivative will be 1, no matter how many times we take the derivative. Thus we need:

$$\begin{aligned} p(0) &= 1 \\ p'(0) &= 1 \\ p''(0) &= 1 \\ &\vdots \end{aligned}$$

Plugging this information in gives us that $c_0 = 1$, since all of the rest of the terms vanish when plugging in 0 into the polynomial. Furthermore $p'(x) = c_1 + 2c_2x + 3c_3x^2 + \ldots$ so plugging in 0 into this and setting it equal to one leaves us with $c_1 = 1$. Taking the second derivative leaves us with $2c_2 + 6c_3x + 12c_4x^2 + \ldots$ and so, again, when $x = 0$ we are left with just the c_2 term and so we have $2c_2 = 1 \implies c_2 = \frac{1}{2}$. In general, we see that because of the power rule and the fact that when $x = 0$ all higher terms cancel, we have that

$$p^{(n)}(0) = n!c_n$$

and so since we want $p^{(n)}(0) = 1$, we have that $c_n = \frac{1}{n!}$ and so our function actually looks like the following:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

We observe on desmos how well this can approximate e^x. The first three terms are $1 + x + \frac{x^2}{2}$, the first three terms are $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

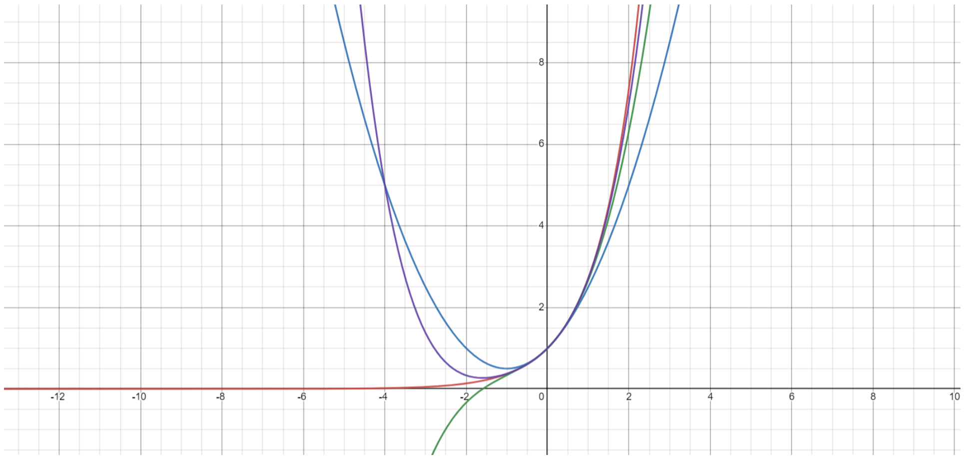

FIGURE 1. The approximations get closer and closer! (Red curve is e^x)

The purple (really close) line is the first 5 terms of the polynomial, the green (still pretty close) is the first 4 terms and so on. So really your calculator is just adding up the first 10-15 terms of this polynomial when you plug in e^x for some x in your calculator. Pretty neat, eh.

7.3. **Generalising This Process.** Now, considering that this is maths society, we want a general formula to do this process with any function f (which isn't a polynomial). Again, let us say that we want to write it in the form $p(x) = \sum_{n=0}^{\infty} c_n x^n$. Then again we would take derivatives and set them equal to the derivative of the original function, except we need not take the derivative at 0, we can pick any point a (for example if we were to do this process with $\ln(x)$, we couldn't pick $x = 0$ to take our derivative on since it isn't defined there). However, like in the e^x example, we want that when we take the nth derivative at a, all of the higher terms cancel out so instead we shall write:

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

Now, just like last time we take the nth derivative of $p(x)$ at a and we find that:

$$p^{(n)}(a) = n! c_n$$

since we've set $p^{(n)}(a) = f^{(n)}(a)$, we find that $c_n = \frac{p^n(a)}{n!}$ and so we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{p^{(n)}(a)}{n!}(x-a)^n$$

which is called the Taylor series of f.