THE HOMOLOGICAL ALGEBRA LECTURE SERIES NOTES

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1. Introduction

I am glad to announce a new set of notes on homological algebra in a series introducing the subject. The last time I tried to do a series of this form, I failed, but this time I am determined to get through it. We begin by thinking about the historical origins of homological algebra before anything, so we must take a look at algebraic topology where it stems from. Poincare realised that whether a space has "n-dimensional holes" is a kind of connectivity.

Consider a simplicial complex X- X can be decomposed into finitely many simplices where a 0-simplex is a point v_1, a 1-simplex is an edge $[v_i, v_j]$, a 2-simplex a triangle $[v_i, v_j, v_k]$, a 3-simplex a tetrahedron $[v_i, v_j, v_k, v_l]$ and there are higher dimensional versions for n-simplices. Homological algebra helps us to answer the question of whether the boundary of an n simplex can be written as the union of $(n-1)$-simplices. For example when $n = 1$, two points should be the boundary of a 1-simplex (line). Indeed X is called *path connected* if there is a path in X connecting two points a and b. If there is no such path, X is said to have a "0-dimensional hole". An example of a one dimensional hole would be in the punctured plane ($\mathbb{R}^2 - \{0\}$), where a perimeter of a triangle with the origin in its centre is not the boundary of a 2-simplex.

Now lets make this a bit more rigorous. In 1, we define the boundary

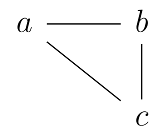

FIGURE 1. The triangle $[a, b, c]$

as follows:

$$\begin{aligned}\partial([a,b,c]) &= [b,c] \cup [c,a] \cup [a,b] \\ &= [b,c] \cup -[a,c] \cup [a,b]\end{aligned}$$

where the $-$ sign comes from the fact that these sides are oriented, so $[a, c] = -[c, a]$. Now, let's just use additive notation for convinience, so we may write:

$$\partial([a,b,c]) = [b,c] - [a,c] + [a,b] \tag{1}$$

Defining the boundary of a 1-simplex $\partial([a, b]) = b - a$ leads to a key observation: *the boundary, when applied twice, gives zero*:

$$\begin{aligned}\partial(\partial([a,b,c])) &= \partial([b,c] - [a,c] + [a,b]) \\ &= (c-b) - (c-a) + (b-a) \\ &= 0\end{aligned}$$

Now let's formalise this for the general case. Form the free abelian group $C_n(X)$ of all linear combinations of n-simplices on X. Then we can define the boundary map:

Definition 1.1. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by:

$$\partial_n([v_0, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

We shall now verify that the boundary map squares to zero- $\partial^2 = 0$.

Proposition 1.1.

$$\partial_{n-1}\partial_n = 0$$

Proof. We have:

$$\begin{aligned}\partial_{n-1}([x_0, \ldots, \hat{x}_i, \ldots, x_n) = &\sum_{j<i} (-1)^j [x_0, \ldots, \hat{x}_j, \ldots \hat{x}_i, \ldots, x_n] \\ &+ \sum_{j>i} (-1)^{j-1} [x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots x_n]\end{aligned}$$

since when $j > i$, we have that x_j will be the $j-1$th term whereas when $j < i$, it'll be the jth. Having put the sums in this form, we see that they are simply the negatives of each other so they cancel to get zero. □

Thus we immediately know that $\mathrm{im}(\partial_n) \subseteq \ker(\partial_{n-1})$.

Definition 1.2. We define the *group of n-cycles* $Z_n(X)$ to be $\ker(\partial_n)$ and the *group of n-boundaries* B_n to be $\mathrm{im}(\partial_{n+1})$.

Thus we can study the following chain of groups:

$$\cdots \to C_3(X) \overset{\partial_3}{\to} C_2(X) \overset{\partial_2}{\to} C_1(X) \overset{\partial_1}{\to} C_0(X)$$

with the property that $\partial^2 = 0$. And so we are well within our rights to study the *simplicial homology groups* of X, defined as:

$$H_n(X) = \frac{Z_n(X)}{B_n(X)}$$

Therefore, if $H_n(X) = 0$, then we know that the space has no n-dimensional holes. [1] As a counter-example, consider the punctured plane X. Then $H_1(X) \neq 0$. This is because if we have $[a, b, c]$ being a triangle with the hole in its interior, then $\alpha = [b, c] - [a, c] + [a, b]$ is a 1-cycle that is not a boundary, for $\alpha = \partial([a, b, c])$ but $[a, b, c]$ is not a 2-simplex in X. Thus the coset $\alpha + B_1(X)$ represents a 1-dimensional hole. Topologists have since modified this construction by adding coefficients by tensoring the complex by some group G or by taking $\mathrm{Hom}(\cdot, G)$, they may compute the cohomology. Throughout this series we shall look at tools for doing such constructions, ensuring a topological focus is maintained throughout.

[1] In general, we would like to attach a meaning to $H_n(X) = 0$ and then we may view the elements of the homology group as obstructions to this property.