An Amazing Connection Between the Riemann Hypothesis and Topology

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The Plan...

1. The Riemann Hypothesis
2. A Number Theoretic Simplicial Complex and its Connection to RH
3. Asymptotic Behaviour of $\beta_k(\Delta_n)$
4. A Couple of Number Theoretic Consequences
5. What Have we Done?

What does it say?

- The zeta function is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1$$

The zeta function admits an analytic continuation to all of $\mathbb{C}$ (except 1). More precisely:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

- The zeta function has trivial zeros at $s = -2n$ because of the sin factor.
- RH asserts that any other zeros of the zeta function are of the form $\frac{1}{2} + it$.

The Mertens Function

- Of great importance to this video is a certain function called the Mertens function:

$$M(n) = \sum_{k=1}^{n} \mu(k)$$

  where μ denotes the Mobius function.
- The importance of this function lies in the rate of it's growth, connecting to to of the biggest number theory problems:
  - PNT $\iff |M(n)| < \epsilon n \,, \epsilon > 0, n \gg 0.$
  - RH $\iff |M(n)| < n^{\frac{1}{2}+\epsilon} \,, \epsilon > 0, n \gg 0$
- But perhaps the most incredible thing is that the Mertens function is, up to a sign, the Euler characteristic of a special simplicial complex!

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Set up/ Motivation

Recall that:

$$\mu(n) = \begin{cases} (-1)^{|P(n)|} & \text{if } n \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$$

Where $P(n)$ is the set of prime factors of n.

- The Euler characteristic of a simplicial complex is $\sum(-1)^n k_n$ where k_n denotes the number of n-simplices.
- Our goal is to construct a simplicial complex that has an Euler characteristic which is $M(n)$ (up to sign) and then study it with topological methods.

Defining the simplicial complex

Defining:

$$\Delta_n = \{P(k) : k \text{ is squarefree and } k \leq n\}$$

gives us the abstract simplicial complex we require.

- Namely, $M(n) = -\chi(\Delta_n)$, where χ is denoting the Euler characteristic minus one.

Betti Numbers

- Betti numbers are a way of rewriting Euler characteristics as another sum. More precisely:

$$\chi(X) = \sum_{k \geq 0} (-1)^k \beta_k(X).$$

- The normal definition would be $\beta_k(X) = \text{rank } H_k(X)$, but since we are dealing with a reduced version of the Euler characteristic, we define $\beta_k(\Delta_n) = \text{rank } \tilde{H}_k(\Delta_n)$.
- Therefore, we have:

$$M(n) = \sum_{k=0}^{\infty} (-1)^{k-1} \beta_k(\Delta_n).$$

- There is a nice, number-theoretic form for these Betti numbers in the case of this complex.

Setting up the nicer form for $\beta_k(\Delta_n)$

Definition

For $x > 0$, define:

- $\sigma(x) = \#\{\text{squarefree integers in } (0, x]\}$
- $\sigma^{odd}(x) = \#\{\text{odd, squarefree integers in } (0, x]\}$
- $\sigma_k(x) = \#\{\text{squarefree integers in } (0, x] \text{ of weight } k\}$
- $\sigma_k^{odd}(x) = \#\{\text{odd, squarefree integers in } (0, x] \text{ of weight } k\}$

Where the weight of a number is the number of prime factors: $\Omega(p_1^{e_1} \dots p_n^{e_n}) = e_1 + \cdots + e_n$ and $\sigma^{even}/\sigma_k^{even}$ are defined analogously.

- A classic approximation for this function is:

$$\sigma_k(x) \sim \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}$$

  which is the prime number theorem for $k = 1$.

- Multiplication by 2 gives a bijection $\sigma_k^{even}(x) = \sigma_{k-1}^{odd}\left(\frac{x}{2}\right)$

Back to the Betti Numbers

The reason I introduced these functions is because of the following result:

Theorem

$$\beta_k(\Delta_n) = \sigma_{k+1}^{odd}(n) - \sigma_k^{odd}\left(\frac{n}{2}\right)$$

Cleaning up the result

Our goal is to get rid of the “odd” part of those σ functions, which we do with the following result:

Theorem

- $\sigma_k^{odd}(x) = \sigma_k(x) - \sigma_{k-1}\left(\frac{x}{2}\right) + \sigma_{k-2}\left(\frac{x}{4}\right) - \sigma_{k-3}\left(\frac{x}{8}\right) + \dots$
- $\sigma^{odd}(x) = \sigma(x) - \sigma\left(\frac{x}{2}\right) + \sigma\left(\frac{x}{4}\right) - \sigma\left(\frac{x}{8}\right) + \dots$

Proof.

We have that $\sigma_k^{even}(x) = \sigma_{k-1}^{odd}\left(\frac{x}{2}\right)$, which means that:

$$\sigma_k(x) - \sigma_{k-1}^{odd}\left(\frac{x}{2}\right) = \sigma_k^{odd}(x)$$

and applying this formula recursively gives the first result. Summation over k gives the second. □

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Motivation

- Recall that our goal is to study the growth of $M(n)$.
- Therefore, studying asymptotic behaviour of $\beta_k(\Delta_n)$ is logical.

The Results:

I shall chose to prove only one of these results, but I will state all of them to give a sense for the type of thing we are dealing with:

Theorem

1. *As $n \to \infty$, we have:*

$$\sum_{k \text{ even}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2} \qquad \text{and} \qquad \sum_{k \text{ odd}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2}$$

2. $\beta_k(\Delta_n) \sim \frac{n}{2\log n}\frac{(\log\log k)^k}{k!}$ *as* $n \to \infty$
3. $\sum_{k=0}^{\infty} \beta_k(\Delta_n) = \frac{2n}{\pi^2} + O(n^\theta)$ *as* $n \to \infty$

Proof of 1

Proof.

Let $a(n)$ and $b(n)$ be the first and second sums respectively. Then 2 and PNT show that

$$\frac{a(n)+b(n)}{n} \to \frac{2}{\pi^2} \qquad \text{and} \qquad \frac{a(n)-b(n)}{n} = \frac{M(n)}{n} \to 0$$

so

$$\frac{2a(n)}{n} \to \frac{2}{\pi^2} \implies a(n) \to \frac{n}{\pi^2}.$$

The proof is analogous for $b(n)$. □

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Overview of what the results will do

- This section is just a quick addendum of some purely number theoretic results that one derives using facts about the Betti numbers of Δ_n
- More precisely, the fact that they are completely determined by the σ functions we defined earlier can help us to find certain bounds.
- Namely, knowing the number of odd, squarefree integers of weight $k + 1$ in the interval $(0, n]$, one can obtain a lower bound for the number of such integers of weight k in the interval $(0, \frac{n}{2}]$, using a couple of preliminary definitions.

Set up

Definition

Given $n, k > 1$, any integer n can be uniquely expressed as

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}.$$

Then we define:

$$\partial_{k-1}(n) \overset{\text{def}}{=} \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_i}{i-1},$$

and

$$\partial^{k-1}(n) \overset{\text{def}}{=} \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1}.$$

The bounds

The final result I shall present comes in two parts:

Theorem

- $\partial_k \left(\sigma_{k+1}^{odd}(n)\right) \leq \sigma_k^{odd}\left(\frac{n}{2}\right)$
- $\partial^k \left(\sigma_{2k+2}^{odd}(n) + \sigma_{2k+1}^{odd}(n)\right) \leq \sigma_{2k}^{odd}(\frac{n}{2}) + \sigma_{2k-1}^{odd}(\frac{n}{2})$

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What we did

- We restated RH as a problem concerning the Euler characteristic of a certain abstract simplicial complex.
- We studied the asymptotic behaviour of the Betti numbers of Δ_n, which was important because $M(n)$ is an alternating sum of those Betti numbers.
- We managed to get some nice bounds for the number of odd, squarefree integers of weight k in the interval $(0, \frac{n}{2}]$, which is a consequence of the work we did earlier, from some slightly messy computations. This is a purely number theoretic result that we managed to obtain using topological methods.

Thanks

Thank you for watching!