Homotopy Groups of Spheres

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The Plan...

The General Slogan for Homotopy Groups of Spheres

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- More precisely, it studies homotopy classes of maps $S^n \to S^k$, relative to basepoints.
- In this talk, I shall talk about the intuition for lots of the basic concepts relating to these homotopy groups.

The Table for Homotopy Groups of Spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

Figure: This is a table of some homotopy groups of spheres. As one can see, it is a huge mess full of different patterns to explore.

The Plan...

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- The fundamental group of the circle can be visualised as wrapping a rubber band around a gluestick.

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- You may twist the rubber band around once, twice, thrice and so forth
- You may do the inverse of this by twisting the rubber band the other way around.

$\pi_1(S^1) \cong \mathbb{Z}$... the image:

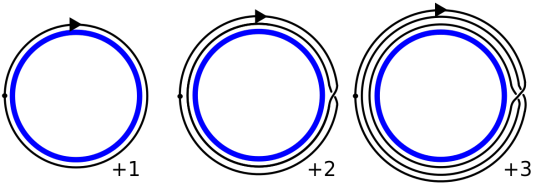

Figure: These are the elements of $\pi_1(S^1)$. Clearly, it is in correspondence with $\mathbb{Z}$, since all of these of these loops can be "unwrapped" by twisting the other way around. Furthermore, two twists of the circle are equivalent if they can be adjusted to each other. The number of times one wraps around the circle is called the winding number.

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- The fundamental group of the sphere can be visualised as a rubber band on a frictionless globe.
- This time, no matter how you place the rubber band on the globe, it can be continuously deformed to a point.
- This means that up to homotopy, every map $S^1 \to S^2$ (which can be visualised as the "wrapping" process described above) is homotopic to a point, so $\pi_1(S^2) = 0$

$\pi_1(S^2) \cong 0$... the image:

Figure: A homotopy, deforming the rubber band to a point.

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- The second homotopy group of the sphere can be visualised as wrapping a football, with some very special wrapping paper.
- The thing that is special about this wrapping football is that when the inside of the wrapping paper touches itself, it dissolves.
- One can wrap the football as many times as they would like- corresponding to the positive branch of the integers. When the friend receives it, he must dissolve each layer of wrapping paper, by wrapping it inside out.

$\pi_2(S^2) \cong \mathbb{Z}$... the image

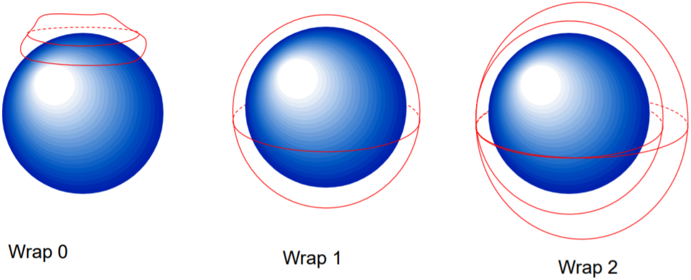

Figure: These are the elements of $\pi_2(S^2)$. Each new integer amount of times you wrap, corresponds to each winding number.

$\pi_2(S^1) \cong 0$

- It is hard to visualise why $\pi_2(S^1) \cong 0$, so I shall just state what $\pi_2(S^1) \cong 0$ actually means.

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- It is hard to visualise why $\pi_2(S^1) \cong 0$, so I shall just state what $\pi_2(S^1) \cong 0$ actually means.
- The second homotopy group of the circle can be visualised as follows: imagine you are now wrapping a ring.
- This time when the friend receives it, he does not need to dissolve it. In fact, it turns out that any way one wraps this ring, the friend can always deform the wrapping paper, without dissolving or cutting it.

$\pi_2(S^1) \cong 0$... the image

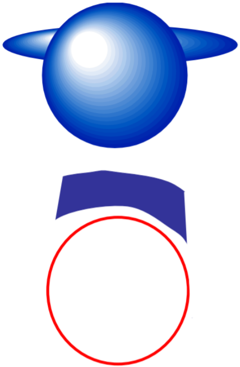

Figure: It turns out that up to homotopy, all wrappings of the sphere around the circle can be shrunk to a point as seen in the image

The Plan...

Warm up: using the same technique to visualise S^2

- First, we want to see how we can see the 2-sphere (just the regular sphere) from the circle. Recall that the equation for the sphere is $x^2 + y^2 + z^2 = 1$. Let us think of the third dimension z as time and refer to it as t. The expression $x^2 + y^2 + t^2 = 1$ can be re-arranged as: $x^2 + y^2 = 1 - t^2$.

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- So really the sphere is just a family of spheres of radius $\sqrt{1 - t^2}$ (by the equation for a circle) as t varies from -1 to 1, where at those points the relation becomes $x^2 + y^2 = 0$ which is just a point.

The image

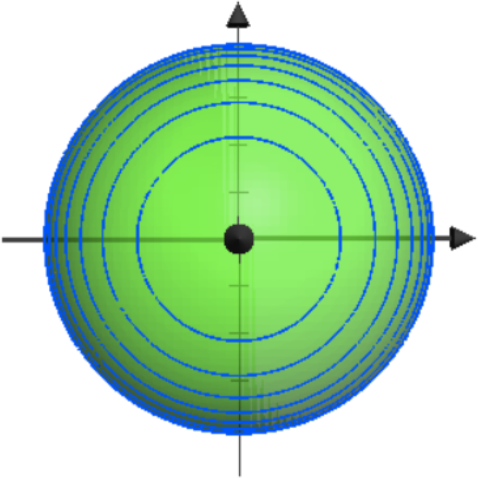


Figure: The sphere, viewed as a family of circles glued in the third dimension.

Warm up: using the same technique to visualise S^1

- Similar to the method outline above, we can see the circle as just a family of 0-spheres glued together in two dimensions. Recall the equation for a circle: $x^2 + y^2 = 1$.

Warm up: using the same technique to visualise S^1

- Similar to the method outline above, we can see the circle as just a family of 0-spheres glued together in two dimensions. Recall the equation for a circle: $x^2 + y^2 = 1$.
- Letting our second dimension y be time, we can rearrange it to be $x = \sqrt{1 - t^2}$. Note that when $t = 0$, this set is precisely $x = \pm 1$, the 0-sphere. As t (remember this is just the y-axis) gets closer to 1, our value $|x|$ gets smaller until $t = \pm 1$, where $|x| = 0$. Overall, all of these co-ordinates (x, t) will look exactly like a circle!

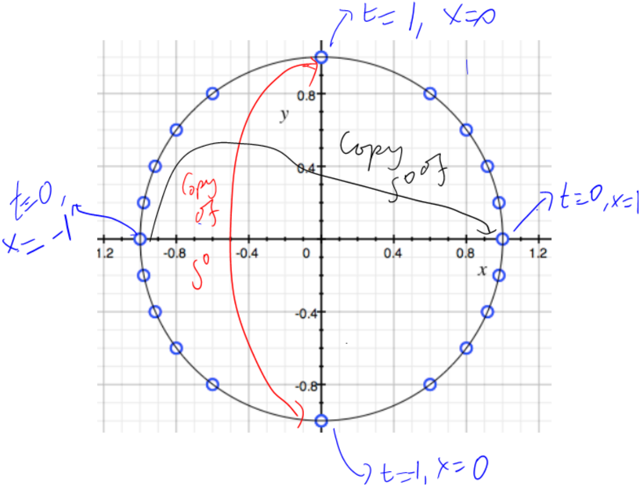
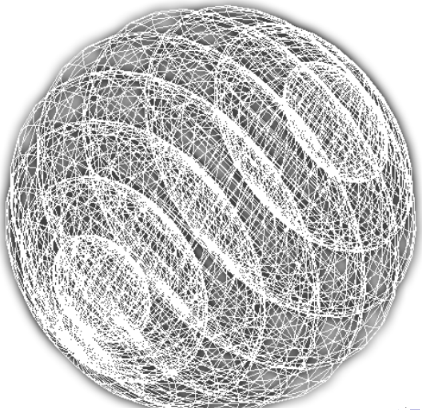


Figure: The circle, viewed as a family of 0-spheres glued together in a second dimension.

Visualising S^3 in this way

- Now comes the fun part- visualising a family of 2-spheres (regular spheres) glued together in a fourth dimension to make something called the 3-sphere (or hypersphere). In the end, we get something that looks amazingly cool:



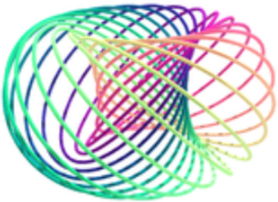
Introduction

Historically, the problem of computing homotopy groups of spheres was suspected to be very simple with the prevailing opinion being that homotopy groups of spheres would be analogous to their homology groups, which were easily computed. but mathematician Heinz Hopf found a counter example via his famous Hopf fibration[1] $S^3 \to S^2$, which can be thought of as a way of somehow "wrapping" the 3-sphere around the 2-sphere.

[1] The Hopf fibration also finds applications in physics, where it models a particle which carries magnetic charge.

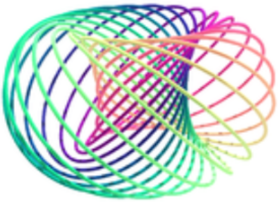
What happens to circles?

- We shall look at this map by slowly unravelling what is happening to different components of the 3 sphere.



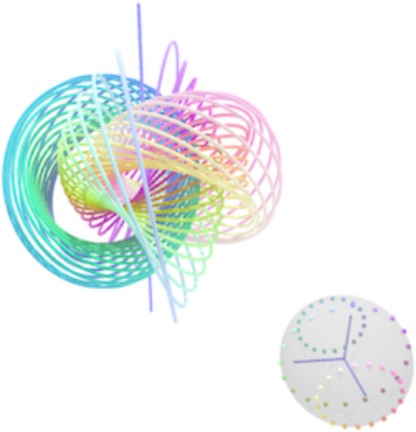
What happens to circles?

- We shall look at this map by slowly unravelling what is happening to different components of the 3 sphere.
- Firstly, it will map circles in a 3-sphere to points on the sphere. The points on the equator are mapped to a torus that one can flip upside down:



What happens on multiple circles?

- Here are how three circles on a 2-sphere correspond via the Hopf map to three linked tori in the 3-sphere:

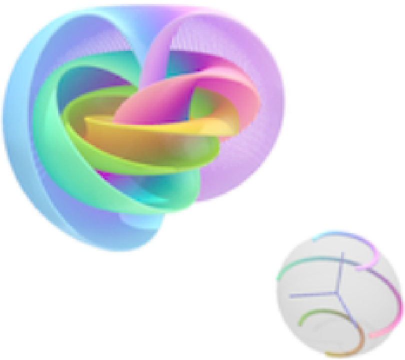


When the fibers form an arc

- Now we see what happens when the fibers form an arc. They actually correspond to what's known as a Hopf link which is an annulus whose boundary circles are linked. They look like this:

The typical energy... when the fibers form a lot of links

When we have the fibers form a lot of links, we get the typical image for the Hopf fibration, in all of its glory:



Technical advantages of the Hopf fibration and other technical remarks

Questions...

I shall pause here for any questions.

The Plan...

Introduction

- The idea behind stable homotopy groups of spheres is that the homotopy groups do indeed stabilise. This means that if we keep going further and further up in dimensions, the homotopy groups will stabilise and eventually settle down completely. We shall look at the example of $\pi_4(S^3)$ later.

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- More precisely, there is a range when $\pi_{n+k}(S^n)$ is independent of n.
- This theorem is a special case of the so called Freudenthal suspension theorem.

The Freudenthal Suspension for spheres

Theorem

There is a homomorphism called the "suspension homomorphism"

$$\pi_{n+k}(S^n) \xrightarrow{\Sigma} \pi_{n+k+1}(S^{n+1})$$

which is an isomorphism if $n \geq k+2$ *and a surjection when* $n = k+1$

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Example

When $k = 1$, we get the following chain of isomorphisms:

$$\pi_4(S^3) \cong \pi_5(S^4) \cong \pi_6(S^5) \cong \dots$$

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- These groups are more well studied than the unstable ones because they are more well behaved.
- To finish, I shall try to convince you that it is the case that $\pi_4(S^3) \cong \mathbb{Z}_2$ (which can be proven rigorously using spectral sequences).

Warm up: intuition for $\pi_3(S^2) \cong \mathbb{Z}$

- Firstly, we shall give intuition for why $\pi_3(S^2) \cong \mathbb{Z}$, before moving up a dimension and investigating $\pi_4(S^3)$.

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- Most points p in S^2 have the property that the points x in S^3 with $f(x) = p$ form a bunch of knots in S^3, which we can think of as a "link". When we pick two different points in S^2 with that property, the links determine an integer called the Hopf invariant.

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- We can count the integer amount of times that these links overlap (with signs depending on whether the links cross over or under each other). This number turns out not to depend on how we picked the two points; it only depends on the homotopy class of f. This is the Hopf invariant and it is bijection with $\pi_3(S^2)$ (that is, $\pi_3(S^2) \cong \mathbb{Z}$).

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- The punchline is that we can undo links in higher dimensions!

Moving up a dimension- $\pi_4(S^3) \cong \mathbb{Z}_2$

As stated in the last slide, links can be undone in 4+ dimensions. So if we compute the Hopf invariant of a map $S^4 \to S^3$ in the same way, then a link can be undone, so the "Hopf invariant" defined the slide before is only defined mod 2 (has two elements).

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- The same reasoning works for all $\pi_{n+1}(S^n)$ (provided that $n \geq 3$)
- This example demonstrates how stable phenomona occur in these situations, because the homotopy groups will eventually agree as we climb the dimensions.

The Plan...

Spectral Sequences

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- Spectral sequences originally were created to compute homology groups by "approximating them with similar spaces", but connections between homotopy and homology is a classic thing to see in algebraic topology so it is no surprise that spectral sequences appear in this setting too.

Bordism

Thinking more geometrically, we have some nice constructions coming from (co)bordism that were mainly developed by Thom and Pontryagin.

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- Soon after, Thom came along and generalised this result using Thom spaces with somehing known as the Thom-Pontryagin construction.

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- The J-homomorphism can be thought of as comparing homotopy groups of spheres to others, and getting certain relationships to exploit.
- Adams' approach using something called K-theory took the lead in determining the image of the J-homomorphism and has now given us information about homotopy groups of spheres.

What good is all this?

I don't know much physics, but there is an interesting discussion on MO which answers the question nicely here.

Thanks

Thanks for watching! Questions?