A Survey of Spectral Sequence Computations

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Contents

1 Introduction

1.1 Motivation

Definition 1.1. Recall that, given a space X, we may form the *cohomology ring* of X:

$$H^*(X) = \bigoplus_n H^n(X). \tag{1}$$

The cohomology ring is graded, by the cap product, which means that there is a product:

$$\smile : H^n(X) \times H^m(X) \to H^{n+m}(X). \tag{2}$$

Furthermore, the cup product is commutative up to a sign: $xy = (-1)^{|x||y|}yx$.

Our goal is to compute this cohomology ring. That is not an easy task. The idea of spectral sequences is to use a lot of easier bits and pieces to compute and then put them all together to obtain $H^*(X)$. To see what I mean, consider the following long exact sequence induced by a CW-pair $A \hookrightarrow X$:

$$\cdots \leftarrow H^n(X) \leftarrow H^n(A) \leftarrow H^n(X, A) \overset{\delta}{\leftarrow} H^{n-1}(X) \leftarrow \dots. \tag{3}$$

But now, say that we can introduce a *filtration* of X. So a nice sequence of inclusions

$$A_0 \hookrightarrow A_1 \hookrightarrow X. \tag{4}$$

This will now give you two long exact sequences:

$$\cdots \leftarrow H^n(A_0) \leftarrow H^n(A_1) \leftarrow H^n(A_0, A_1) \overset{\delta}{\leftarrow} H^{n-1}(A_0) \leftarrow \dots. \tag{5}$$

and

$$\cdots \leftarrow H^n(X) \leftarrow H^n(A_1) \leftarrow H^n(X, A_1) \overset{\delta}{\leftarrow} H^{n-1}(X) \leftarrow \dots. \tag{6}$$

The idea now, is that the cohomology of A_1 and A_0 should be easier to compute than $H^*(X)$. After doing the computation of them, we can hope to move up to the cohomology of X. However, we need not stop at 2 exact sequences! Let's say that we can introduce a filration which looks like this:

$$A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n = X. \tag{7}$$

This now gives rise to many long exact sequences, each approximating the cohomology of X. We can then try to use all of the data we have collected to move up to the cohomology of X. This data comes together to make a spectral sequence.

1.2 Precise Definitions

Definition 1.2. A spectral sequence $\{E_r^{p,q}, d_r\}$ consists of the following information:

- For all $p, q, r > 0$, $E_r^{p,q}$ is an abelian group.
- The differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfy $d^2 = 0$
- The homology of the rth page is the $r+1$th page. That is: $H(E_r^{p,q}; d_r) = E_{r+1}^{p,q}$.

We may visualise some of the pages, where the dots are groups:

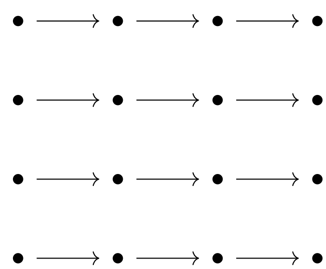


Figure 1: The E_1 page- the differentials go across the p-axis by one and don't go up the q axis since $-1+1=0$

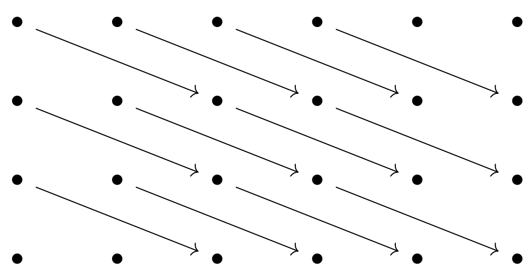


Figure 2: This is the E_2 page; each dot is a group, and each arrow is a differential d_2, moving across the p axis by 2 and down the q axis by 1.

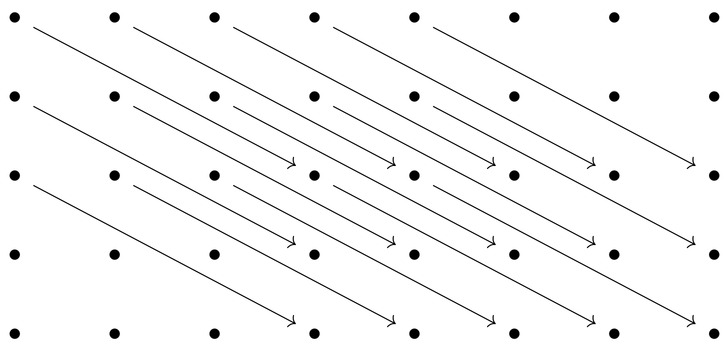


Figure 3: The E_3 page-hopefully you see the pattern now.

It is often the case that E_1 is a well known complex, so normally we state the E_2 page when starting, being the standard cohomology of E_1.

Definition 1.3. If there is some $r \gg 0$ such that all of the differentials $d_{r'}$ are zero for $r' \geq r$, we say that

$$E_r = E_\infty \tag{8}$$

since

$$E_{r+1} = H(E_r) = E_r. \tag{9}$$

Furthermore, if there is a graded object H_* such that summing along the diagonals gives H_n;

$$H_n = \bigoplus_{p+q=n} E_\infty^{p,q}. \tag{10}$$

If this happens, we say that the spectral sequence converges to H_* and write:

$$E_2^{p,q} \implies H_* \tag{11}$$

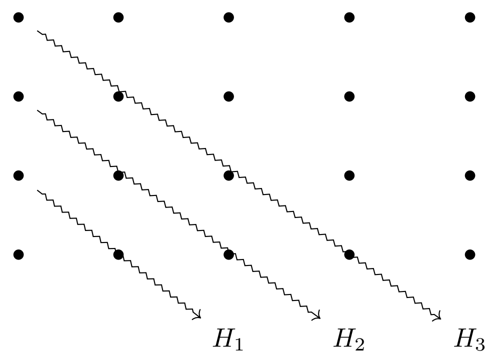


Figure 4: Summing along the diagonals $p+q=1$, $p+q=2$, $p+q=3$ yields H_1, H_2, H_3 respectively, along the E_∞ page.

Part I

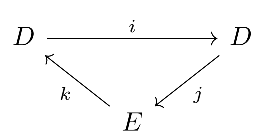
Spectral Sequences in Topology

2 Bockstein Spectral Sequence

2.1 Exact Couples

We begin this section by introducing the notion of an exact couple in order to set up the Bockstein spectral sequence, which is a setting in which spectral sequences arise very naturally.

Definition 2.1. An *exact couple*, (D, E, i, j, k) consists of two modules, along with three morphisms $i : D \to D$, $j : D \to E$ and $k : E \to D$ such that:



(12)

is exact at each vertex.

Note that if we set $d = jk : E \to E$ it becomes a differential, since $d^2 = jkjk = j(kj)k = 0$. We now define the *derived couple* as follows:

Definition 2.2. Given an exact couple, of the form in 12, we define a new exact couple (D', E', i', j', k') with:

- $D' = i(D)$
- $E' = H_*(E; d)$

- $i' = i|_{D'}$
- $j'(i(x)) = j(x) + jk(E) = [j(x)]$
- $k'([y]) = k(y)$

Proposition 2.1. The derived couple is exact

Proof. Exercise □

Because of this, we may do this again, taking the derived couple again, leaving us with a lot of derived couples: $(D^r, E^r, i^r, j^r, k^r)$.

- $(D^1, E^1, i^1, j^1, k^1) = (D, E, i, j, k)$
- $(D^2, E^2, i^2, j^2, k^2) = (D', E', i', j', k')$ and so forth...
- The differential d is given by $d^r = j^r k^r$

turns (E^r, d^r) into a spectral sequence!

2.2 The Bockstein Spectral Sequence Construction

Now, we may construct the Bockstein specral sequence.

3 Serre Spectral Sequence

3.1 $H^*(S^2)$

Proposition 3.1.

$$H^*(S^3) \cong \mathbb{Z}[x]/(x^2),\ |x| = 3 \tag{13}$$

Proof. **Step 1:** Determine the nontrivial elements on the E_2 page and differentials. For S^2, this is simple via the Hopf fibration:

$$S^1 \to S^3 \to S^2. \tag{14}$$

Now that we know which fiber bundle we're working with, we may determine the E_2 page. More precisely:

$$E_2^{p,q} = H^p(S^2; H^q(S^1)) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 2 \text{ and } q = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Now time for some pictures:

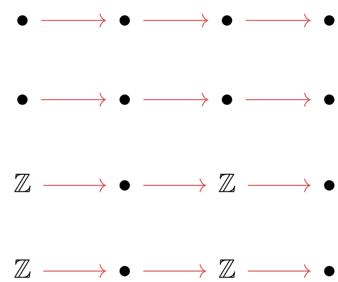


Figure 5: Every differential on the E_1 page is trivial

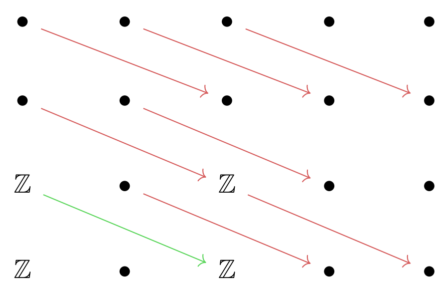


Figure 6: The differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ is nontrivial.

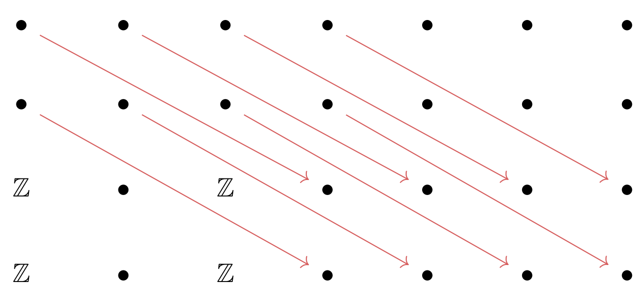


Figure 7: For $r \geq 3$, all of the differentials d_r will have too high of a degree to be nontrivial. Therefore $E_3 = E_\infty$.

Next we show that the differential $d_2 : E_2^{0,1} \to E_2^{2,0}$. is an isomorphism, given that:

- $H^0(S^3) = \mathbb{Z}$
- $H^1(S^3) = 0$
- $H^2(S^3) = 0$
- $H^3(S^3) = \mathbb{Z}$

we have that $d_2(1) = \pm 1$. So if $x \in E_2^{0,1}$ is a generator, then $d_2(x) \in E_2^{2,0}$ is also a generator. Furthermore, setting $y = d_2(x)$, we have that:

- x generates $E_2^{0,1}$
- y generates $E_2^{2,0}$
- xy generates $E_2^{2,1}$
- $y^n = 0$, $n \geq 2$
- $x^n = 0$, $n \geq 2$

I claim that d_2 is an isomorphism. **Show that d_2 is an isomorphism:**

$\square$

3.2 $H^*(\mathbb{CP}^\infty)$

Theorem 3.2. *$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[x]$, $|x| = 2$.*

Proof. We will use the fiber bundle:

$$S^1 \to S^\infty \to \mathbb{CP}^\infty \tag{16}$$

for the setup of the SSS. We see which terms on the E_2 page are nontrivial:

$$E_2^{p,q} = H^p(\mathbb{CP}^\infty); H^q(S^1)) = \begin{cases} H^p(\mathbb{CP}^\infty) & \text{if } q = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Picture time:
insert da pic

Since the spectral sequence converges to $H^*(S^\infty) = 0$, the only nontrivial term on the E_∞-page will be $E_\infty^{0,0}$. We will now figure out some generators, and try to use induction to compute every $H^n(\mathbb{CP}^\infty)$. The terms we shall observe are:

- $E_2^{1,0}$
- $E_2^{0,1}$
- $E_2^{2,0}$

- $E_2^{2,1}$

Firstly: $E_2^{1,0}$. For each $r \geq 2$, we have it that $d_r : E_2^{1,0} \to E_2^{1+n,-r+1}$ go into a trivial group, and $d_r : E_2^{*,*} \to E_2^{1,0}$ go from a trivial group. This implies that $E_2^{1,0} = E_\infty^{1,0}$ which, since $E_\infty^{1,0} \neq E_\infty^{0,0}$, is trivial. The only differential going into $E_2^{2,0}$ is

$$d_2 : E_2^{0,1} \to E_2^{2,0}. \tag{18}$$

Similarly d_2 is the only nontrivial differential going into $E_{2,0}^2$. We would like to show that d_2 is an isomorphism. Since

$$0 = E_\infty^{0,1} = E_3^{0,1} = \frac{\ker d_2 : E_2^{0,1} \to E_2^{2,0}}{\text{im } d_2 : E_2^{-2,1} \to E_2^{0,1}}. \tag{19}$$

In order to make this true, $\ker d_2 = 0$, so d_2 is an injection. Similarly,

$$0 = E_\infty^{2,0} = E_3^{2,0} = \frac{E_2^{2,0}}{\text{im } d_2 : E_2^{0,1} \to E_2^{2,0}} \tag{20}$$

so im $d_2 = E_2^{2,0}$ so it must be a surjection. Therefore

$$E_2^{2,0} \cong E_2^{0,1} \cong \mathbb{Z}. \tag{21}$$

Similarly, one can show that $E_2^{2n-1,0} H^{2n-1}(\mathbb{CP}^\infty) \cong 0$ and $E_2^{2n,0} = H^{2n}(\mathbb{CP}^\infty) \cong \mathbb{Z}$. Now pick a generator x for $E_2^{0,1}$. Then:

- $d_2(x) = y \in E_2^{2,0}$ is a generator.
- $xy \in E_2^{2,1}$ is a generator.
- $d_2(xy)$ generates $E_2^{4,0}$.

However, remember that d_2 is multiplicative in the sense that:

$$d_2(xy) = d(x)y + d(y)x = y^2 + 0 = y^2. \tag{22}$$

So y^2 generates $E_2^{4,0} = H^4(\mathbb{CP}^\infty)$. Continuing this process gives the following information:

- $H^{2n-1}(\mathbb{CP}^\infty) = 0$
- $H^{2n}(\mathbb{CP}^\infty) = \mathbb{Z}$, generated by y^n.

So overall:

$$\boxed{H^*(\mathbb{CP}^\infty) = \mathbb{Z}[x],\ |x| = 2} \tag{23}$$

□

3.3 $H^*(\Omega S^3)$

Proposition 3.3.

$$H^*(\Omega S^3) \cong \Gamma[x],\ |x| = 2 \tag{24}$$

Proof. The fiber bundle we shall use is:

$$\Omega S^3 \to PS^3 \simeq * \to S^3. \tag{25}$$

Remarks:

- We know the cohomology of S^3 and $PS^3 3$, so we can go ahead and use the Serre spectral sequence.
- Since $PS^3 \simeq *$, we have it that $H^*(PS^3) \cong 0$, so the only nontrivial element of the E_∞ page is $E_\infty^{0,0}$. This is because our spectral sequence will converge to $H^*(PS^3)$.

□

We shall now take a closer look at the E_2 page:

$$E_2^{p,q} = H^p(S^3; H^q(\Omega S^3)) = \begin{cases} H^q(\Omega S^3) & \text{if } p = 0, 3 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

So the only differential that we care about is d_3.

3.4 $H^*(SU(n))$

Proposition 3.4.

$$H^*(SU(n)) \cong \Lambda[x_3, \dots x_{2n-1}] \tag{27}$$

where $|x_n| = n$

Proof. We shall use induction. Thankfully, the base case is very simple since when $n = 2$, we have: $SU(2) \cong S^3$ so

$$H^*(SU(2)) \cong H^*(S^3) \cong \Lambda[x_3] \tag{28}$$

Now assume that $H^*(SU(n-1)) \cong \Lambda[x_3, x_5, \dots x_{2n-3}]$. This means that in the fiber bundle:

$$SU(n-1) \to SU(n) \to S^{2n-1}, \tag{29}$$

we know the cohomology of $SU(n-1)$ and S^{2n-1}, so we are in position to use the Serre spectral sequence. Let's take a look at the E_2 page:

$$E_2^{p,q} = H^p(S^{2n-1}; H^q(SU(n-1))) = \begin{cases} H^q(SU(n-1)) & \text{if } p = 0, 2n-1 \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

The argument for this proof is to show that the spectral sequence will collapse at the E_2 page. You might be thinking that this can't be the case, since the $0th$ and $(2n-1)th$ columns are interesting, however it turns out that all of the differentials d_{2n-1} will be trivial. Since the differentials are all of bidegree $(2n-1, -2n-2)$, it's clear that $d_{2n-1}(a_i) = 0$. The only ones left for the killing are the groups generated by multiple of these generators. However, the multiplicative structure takes care of that:

$$d_{2n-1}(a_i a_n) = d_{2n-1}(a_i)a_n \pm d_{2n-1}(a_n)a_i = 0 \pm 0 = 0. \tag{31}$$

So clearly, all the differentials on the E_{2n-1} page are trivial, and $E_2 = E_\infty$. So now, summing along the diagonals of the E_2 page leaves us with

$$H^*(SU(n)) \cong \Lambda[x_3, \ldots x_{2n-1}] \tag{32}$$

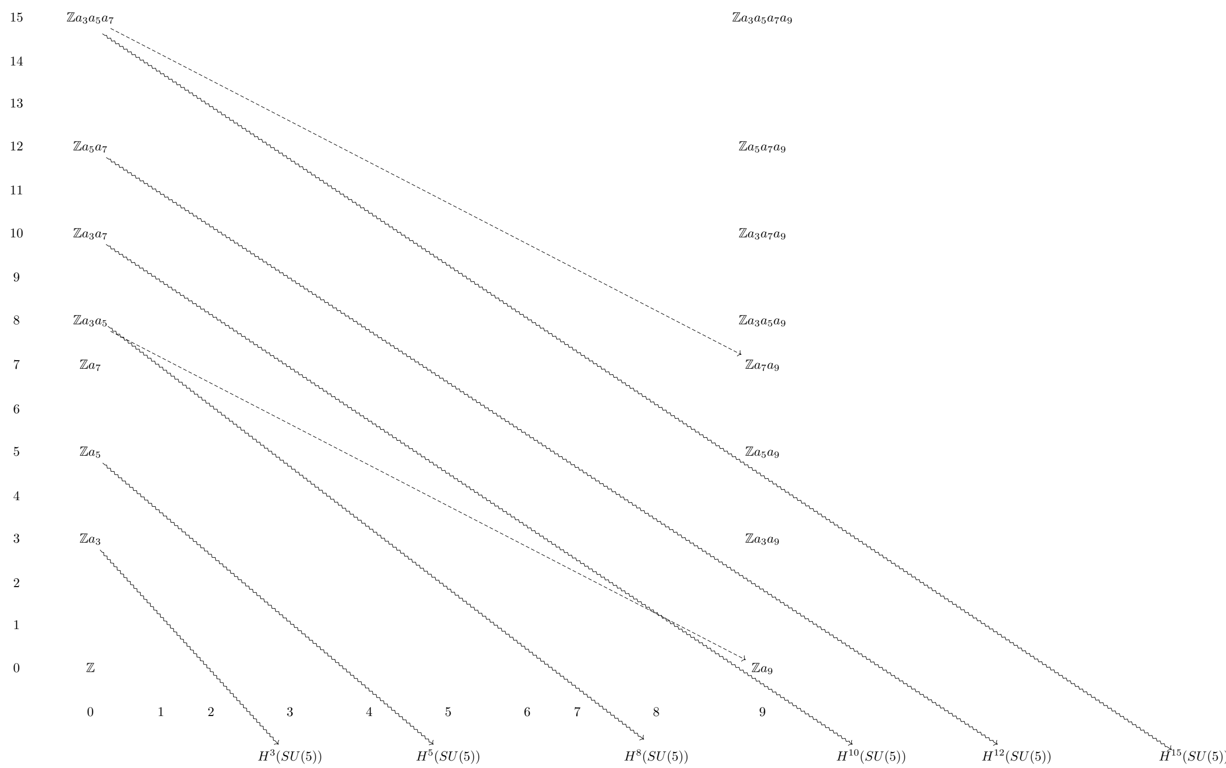


Figure 8: Deducing the cohomology of $SU(5)$ by summing along the diagonals of the E_2 page, assuming $H^*(SU(4)) \cong \Lambda[x_3, \ldots x_7]$

□

3.5 $\pi_1^s(S^0)$

We now consider some applications to homotopy theory. To do so we require a little bit more set up:

Theorem 3.5. *If X is simply connected, and $H_k(X) = 0$ for $0 < k < n$, then there is a map inducing an isomorphism on homotopy groups:*

$$F \to X \to K(\pi_n(X), n), \tag{33}$$

where F is the homotopy fiber such that

$$\pi_i(F) = \begin{cases} 0 & \text{if } k \leq n \\ \pi_i(X) & \text{if } k > n \end{cases} \tag{34}$$

Corollary 3.6 (The Upshot)**.** By the Hurewicz theorem, we see that

$$H_{k+1}(F) = \pi_{n+1}(F) = \pi_{n+1}(X). \tag{35}$$

Hence, if we get such a fiber bundle, we can apply our findings to homotopy theory! A brilliant example of this is computing the first stable stem, aka $\pi_4(S^3)$. Our set up will be of the form:

$$F \to S^3 \to K(\mathbb{Z}.3) \tag{36}$$

which, up to homotopy, gives us a fiber bundle:

$$\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) = \mathbb{CP}^\infty \to F \to S^3. \tag{37}$$

Since we know the cohomology of $\mathbb{CP}^\infty$ and S^3, we get something that we may compute with the Serre spectral sequence. Our plan, roughly will look like this:

- Figure out the nontrivial terms in the E_2 page
- Figure out where the spectral sequence collapses.
- Compute the differentials by looking at the generators.
- Collect together the terms on the E_∞ page, to obtain some results related to $H^*(F)$
- Pass to universal coefficients to obtain results related to $H_*(F)$
- Use the setup from above to get: $H_4(F) \cong \pi_4(F) \cong \pi_4(S^3)$.

First, see that the E_2 page looks like:

$$E_2^{p,q} = H^p(S^3; H^q(\mathbb{CP}^\infty)) = \begin{cases} H^q(\mathbb{CP}^\infty) & \text{if } p = 0,3 \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

Furthermore, since $H^q(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & \text{if } q \text{ is even} \\ 0 & \text{otherwise} \end{cases}$
we see that the only nontrivial terms in the E_2 page are:

- $E_2^{0,2n}$
- $E_2^{3,2n}$

or, pictorially,

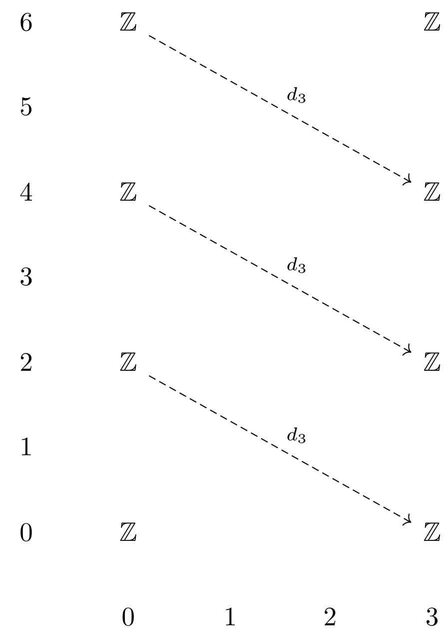


Figure 9: The only nontrivial differentials are $d_3 : E_3^{p,q} \to E_3^{p+3,q-2}$, since both of the degrees work out. Furthermore, we conclude that $E_4 = E_\infty$.

Now we pick some generators, in order to make computing the differentials easier:

- Let $u \in E_3^{3,0}$ generate $H^3(S^3)$.
- Now let $x \in E_3^{0,2}$ generate $H^2(\mathbb{CP}^\infty)$.

We immediately see:

- x^n generates $E_3^{0,2n}$.
- $u^n = 0$, $n > 0$.
- ux^n generates $E_3^{3,2n}$.

Pictorially this can be visualised as follows:

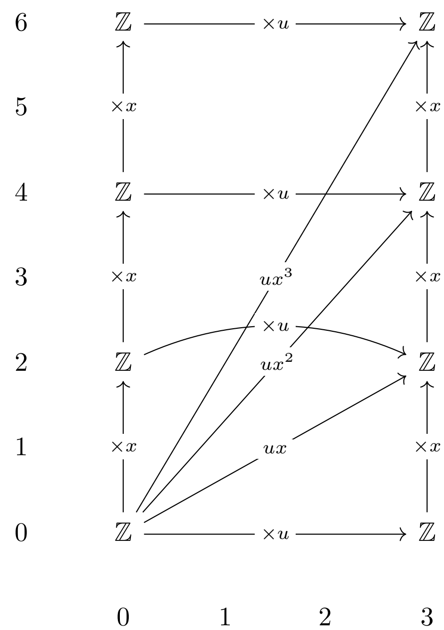


Figure 10: Multiplication by x goes up the ladder, whereas multiplication by u goes across.

We now compute the differential on the generators:

Proposition 3.7. We have

$$d_3(x) = \pm u. \tag{39}$$

Now, since $d_3 : E_3^{0,2n} \to E_3^{3,2(n-1)}$ is a derivation, we have that

$$d_3(x^n) = \pm nux^{n-1}. \tag{40}$$

With this information, we can finally talk about the $E_4 = E_\infty$ page! More precisely:

- $E_4^{0,2n} = H(E_3^{0,2n}; d_3) = \frac{\ker(d_3 : E_3^{0,2n} \to E_3^{3,2(n-1)})}{\operatorname{im}(d_3 : E_3^{-3,2(n+1)} \to E_3^{0,2n})}$. Clearly, since d_3 is an injection, $E_4^{0,2n} = 0$.
- $E_4^{3,2(n-1)} = \frac{\ker(d_3 : E_3^{3,2(n-1)} \to E_3^{6,2(n-2)})}{\operatorname{im}(d_3 : E_3^{0,2n} \to E_3^{3,2(n-1)})}$. Since $E_3^{6,2(n-2)} = 0$, the kernel of the map will be everything (i.e. $\mathbb{Z}$). Furthermore, since the image of d_3 is generated by $\pm nux^{n-1}$, it acts somewhat like multiplication by n. Hence we get

$$E_4^{3,2(n-1)} \cong \mathbb{Z}_n. \tag{41}$$

6				$\mathbb{Z}_4$
5				
4				$\mathbb{Z}_3$
3				
2				$\mathbb{Z}_2$
1				
0				
	0	1	2	3

Figure 11: The E_4 page

When we pick $n = 2$, we have that:

$$E_4^{3,2} = H^5(F) = \mathbb{Z}_2 \tag{42}$$

since the spectral sequence converges to $H^*(F)$. Now we know $H^5(F) = \mathbb{Z}_2$. After dropping torsion a degree, we see that

$$H^5(F) = H_4(F) \tag{43}$$

and so overall we get

$$H_4(F) = \pi_4(F) = \pi_4(S^3) = \mathbb{Z}_2 \tag{44}$$

3.6 $\pi_4(SU(3))$

Theorem 3.8.

$$\pi_4(SU(3)) = 0 \tag{45}$$

Proof. need to figure out :blobabouttocry: □

4 The Atiyah-Hirzebruch Spectral Sequence

4.1 What is it?

Theorem 4.1.

4.2 K-Theory of $\mathbb{CP}^n$

We figure out

4.3 Stable Cohomology Operations and the Nontrivial Differentials

Part II
Derived Categories and the Grothendieck Spectral Sequence

5 Derived Functors

6 Grothendieck Spectral Sequence

7 Improving the Grothendieck Spectral Sequence-Derived Categories