Spectral Sequence Talk

Gregor Sanfey

May 2021

The Plan...

Motivation

Basic Definitions

Serre Spectral Sequence

Brief Interlude: Generalised Cohomology and Spectra

Atiyah-Hirezbruch Spectral Sequence

Our Goal

- X CW cell complex- we want to compute $H^*(X)$.
- H^* is graded, via the cap product.
- However, computing $H^*(X)$ is much easier said than done.

One solution to this problem lies in spectral sequences.

Making the Job Easier...

Assume that $A \hookrightarrow X$ is a CW pair. Then we obtain a long exact sequence in cohomology:

$$\cdots \leftarrow H^n(X) \leftarrow H^n(A) \leftarrow H^n(X,A) \overset{\delta}{\leftarrow} H^{n-1}(X) \leftarrow \dots$$

- This is good, because it helps us to obtain information about $H^*(X)$.
- Yet we need not stop here! We can introduce a filtration- two CW pairs $A_0 \hookrightarrow A_1 \hookrightarrow X$.
- This now breaks down the problem of computing $H^*(X)$ into 2 even smaller pieces.

Filtering the CW Pair Further

- We can continue like this:

$$A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_{n-1} \hookrightarrow X.$$

- This breaks down the problem further
- The algebraic tool used for storing all of the data encoded by the long exact sequences is called a spectral sequence.

So what is a spectral sequence, precisely?

Definition

A spectral sequence is a collection $\{E_r^{p,q}; d_r\}$ such that:

1. $E_r^{p,q}$ is an abelian group for all r, p, q
2. $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ (that is each differential is of degree $(r, -r+1)$) such that $d_r^2 = 0$
3. $H(E_r; d_r) = E_{r+1}$

Visualising the first few pages

The E_∞ page

We often write the E_2 page first since the E_1 page is often a well understood complex already.

- Often, there will be an $r \gg 0$ such that the differentials $d_{r'}$ are trivial for $r' \geq r$, then $E_r = H(E_r; d_r) = E_{r+1} = E_{r+2} = \dots$.
- This page is called the E_∞ page
- We say that a spectral sequence converges to a graded object H^* if we can recover each H^n by summing along the diagonals of the E_∞ page modulo extension problems which we won't encounter in this talk:

$$H^n = \bigoplus_{p+q=n} E_\infty^{p,q}$$

- In this case, we write

$$E_2^{p,q} \implies H^*$$

Our general strategy

Our general strategy will be:

- Compute every page until we hit the E_∞ page
- Recover the homology by summing along the diagonals

What problems will arise?

- What are the differentials?
- When exactly will the spectral sequence collapse?
- We will see what else will cause us problems along the way. . .

What is it?

Theorem

Given a Serre fibration $F \to E \to B$ with simply connected base space, there is a spectral sequence called the Serre spectral sequence of the form:

$$E_2^{p,q} = H^p(B; H^q(F)) \implies H^*(E).$$

Example Computation: $H^*(\mathbb{CP}^\infty)$

In order to understand how using the Serre spectral sequence works, we shall use an example:

Theorem

$$H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x], |x| = 2$$

Figuring Out the E_2 Page

- Recall: $S^1 \to S^\infty \to \mathbb{CP}^\infty$ fibration.

Therefore,

$$E_2^{p,q} = H^p(\mathbb{CP}^\infty; H^q(S^1)) = \begin{cases} H^p(\mathbb{CP}^\infty), & \text{if } q = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

- This spectral sequence converges to $H^*(S^\infty)$, but $S^\infty \simeq *$.
- Therefore the only nontrivial element of the E_∞ page is $E_\infty^{0,0}$.

Visualising the E_2 page:

Which differentials do we care about?

Since the only nontrivial elements of the E_2 page occur when $q = 0, 1$, the only nontrivial differentials will be of the form:

$$E_2^{*,1} \to E_2^{*,0}$$

and that is the d_2 differential. Therefore:

- $E_3 = E_\infty$
- Any element of the E_2 page such that there is no nontrivial differential going to or from it will be trivial.

Searching for generators

We strive to compute:

- $E_2^{2,0}$
- $E_2^{1,0}$

and then use the generators to do everything else for us.

Theorem

$d_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism.

Showing that it is an isomorphism

- For injectivity, it is enough to show that $\ker(d_2) = 0$. To do this, remember that:

$$E_3^{0,1} = \frac{\ker(d_2 : E_2^{0,1} \to E_2^{2,0})}{\mathrm{im}(d_2 : E_2^{-2,2} \to E_2^{0,1})} = 0$$

  Which shows that it is injective since $E_2^{-2,2} = 0$.

- For surjectivity it is enough to show that $\mathrm{coker}(d_2) = 0$. We use the exact same reasoning as before:

$$E_3^{2,0} = \frac{\ker(d_2 : E_2^{2,0} \to E_2^{4,-1})}{\mathrm{im}(d_2 : E_2^{0,1} \to E_2^{2,0})} = \frac{E_2^{2,0}}{\mathrm{im}(d_2 : E_2^{0,1} \to E_2^{2,0})} = 0$$

$E_2^{1,0}$ is trivial

- The differential going to $E_2^{1,0}$ is:

$$d_2 : E_2^{-1,1} = 0 \to E_2^{1,0}$$

- The differential from $E_2^{0,1}$ is:

$$d_2 : E_2^{1,0} \to E_2^{3,-1} = 0$$

Therefore, $E_2^{1,0} = E_\infty^{1,0} = 0$

Nearly there... what information do we have?

- $E_2^{0,1} \cong E_2^{2,0} \cong \mathbb{Z}$. Continuing by induction tells us: $E_2^{2n,0} = H^{2n}(\mathbb{CP}^\infty) \cong \mathbb{Z}$
- $E_2^{1,0} \cong 0$. Using this and induction shows that $E_2^{2n-1,0} = H^{2n-1}(\mathbb{CP}^\infty) = 0$.

Now let y generate $E_2^{0,1}$. Then $d_2(y) = x$ generates $E_2^{2,0}$. Hence:

- xy generates $E_2^{2,1}$
- $d_2(xy)$ generates $E_2^{4,0}$. Yet

$$d_2(xy) = d_2(x)y + d_2(y)x = x^2$$

Continuing by induction shows that x^n generates $E_2^{2n,0} = H^{2n}(\mathbb{CP}^\infty)$ so:

$$\boxed{H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x],\ |x| = 2}$$

Recap:

- We determined where the nontrivial elements of the E_2 page were
- Then, we looked for the nontrivial differentials.
- Then we used the fact that the spectral sequence converges to S^∞ to get more information about the E_2 page.
- Then, with this information, we found some generators and pieced all the information together to get the final result.

Application 1: Hurewicz Theorem

Theorem

If X is $(n-1)$-connected, $n \geq 2$ then $\pi_n(X) \cong H_n(X)$ and $\tilde{H}_i(X) = 0$, $i \leq n-1$

Setting everything up

Before we proceed, we need to see what we're working with here:

- We will apply the Serre spectral sequence to $\Omega X \to PX \simeq * \to X$

The base case

We start off for $n = 2$.

$$\pi_2(X) \cong \pi_1(\Omega X) \cong H_1(\Omega X)$$

- The last isomorphism is the abelianisation, since $\pi_1(\Omega X) = \pi_2(X)$ which is abelian.
- Now we must show that $H_2(X) \cong H_1(\Omega X)$.
- The E_2 page is given by: $E^2_{p,q} = H_p(X; H_q(\Omega X))$

Visualising the E^2 page:

Showing the isomorphism

Theorem

The map $d^2 : E^2_{2,0} = H_2(X) \to E^2_{0,1} = H_1(\Omega X)$

Proof.

Since $PX \simeq *$, we can use the same reasoning as before with our $H^*(\mathbb{CP}^\infty)$ reasoning to show that d^2 must be an isomorphism. □

The Inductive Step:

This time assume the Hurewicz theorem for $n-1$. We show that it is true for n.

- Since X is $(n-1)$-connected, ΩX is $(n-2)$-connected.
- By the hypothesis applied to ΩX, we have that $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$.
- This then implies that $\pi_n(X) \cong H_{n-1}(\Omega X)$.

Now we use the spectral sequence!

- In this case, the E^2 page is interesting because:

$$E^2_{p,q} = H_p(X; H_q(\Omega X)) = 0$$

  when $q < n-1$, by the induction hypothesis on ΩX.

- This means that everybody on the p axis, $p \leq n$ doesn't get affected by the differentials $d^2, \dots d^n$.
- The spectral sequence converges to $PX \simeq *$, so everything has to get killed somehow hence $d^n : E^n_{n,0} = H_n(X) \to E^n_{0,n-1} = H_{n-1}(\Omega X)$ must be an isomorphism and $H_i(X) = 0$, $1 \leq i \leq n-1$.

Computation 2: $\pi_4(S^3)$

Theorem

$$\pi_4(S^3) \cong \mathbb{Z}_2$$

Setting everything up

Before we dive into this proof, we need to somehow use spectral sequences to compute not only homology but homotopy groups too. To do so we use the following theorem:

Theorem

If X is simply connected, and $H_k(X) = 0$ for $0 < k < n$, then there is a map inducing an isomorphism on homotopy groups: $F \to X \to K(\pi_n(X), n)$ where F is the homotopy fiber such that

$$\pi_i(F) = \begin{cases} 0 & \text{if } k \leq n \\ \pi_i(X) & \text{if } k > n \end{cases}$$

Corollary

By the Hurewicz theorem, we see that:
$H_{k+1}(F) = \pi_{n+1}(F) = \pi_{n+1}(X)$.

Applying this to S^3:

Our strategy will be:

- Apply the theorem to get a fibration $F \to S^3 \to K(\mathbb{Z}, 3)$ and hence, up to homotopy, get another fibration: $\Omega K(\mathbb{Z}, 3) \simeq K(\mathbb{Z}, 2) = \mathbb{CP}^\infty \to F \to S^3$
- Use the strategies as before to obtain $H^5(F)$
- One can drop torsion a degree and hence obtain $H^5(F) \cong H_4(F) \cong H_4(S^3) \cong \pi_4(S^3)$.

The E_2 page

First we obtain $E_2^{p,q}$:

$$E_2^{p,q} = H^p(S^3; H^q(\mathbb{CP}^\infty)) = \begin{cases} H^q(\mathbb{CP}^\infty) & \text{if } p = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, $H^q(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & \text{if } q \text{ is even} \\ 0 & \text{otherwise} \end{cases}$, so the only nontrivial terms of the E_2 page are:

- $E_2^{0,2n}$
- $E_2^{3,2n}$

Visualising the E_2 page

Where Does it Collapse?

- Clearly, the differentials d_2 don't go far enough across to be nontrivial
- The d_3 differentials, however, do the job.
- For $n \geq 4$, the differentials d_4 go too far, so the spectral sequence collapses at the E_4 page.

What are the nontrivial differentials?

First, pick:

- Let u generate $E_3^{3,0}$
  - $u^n = 0,\ n > 1$
- Let x generate $E_3^{0,2}$
  - x^n generates $E_3^{0,2n}$
- ux^n generates $E_3^{3,2n}$

Now we can "compute the differentials without computing the differentials":

$$d_3(x) = \pm u$$

The E_4 page:

Now, since $d_3 : E_3^{0,2n} \to E_3^{3,2(n-1)}$ is a derivation, we have that

$$d_3(x^n) = \pm nux^{n-1}.$$

With this information, we can finally talk about the $E_4 = E_\infty$ page! More precisely:

- $E_4^{0,2n} = H(E_3^{0,2n}; d_3) = \frac{\ker(d_3 : E_3^{0,2n} \to E_3^{3,2(n-1)})}{\operatorname{im}(d_3 : E_3^{-3,2(n+1)} \to E_3^{0,2n})}$. Clearly, since d_3 is an injection, $E_4^{0,2n} = 0$.
- $E_4^{3,2(n-1)} = \frac{\ker(d_3 : E_3^{3,2(n-1)} \to E_3^{6,2(n-2)})}{\operatorname{im}(d_3 : E_3^{0,2n} \to E_3^{3,2(n-1)})}$. Since $E_3^{6,2(n-2)} = 0$, the kernel of the map will be everything (i.e. $\mathbb{Z}$). Furthermore, since the image of d_3 is generated by $\pm nux^{n-1}$, it acts somewhat like multiplication by n. Hence we get

$$E_4^{3,2(n-1)} \cong \mathbb{Z}_n.$$

Visualsing the E_∞ page

Collecting the information

- Let's pick $n = 2$. This gives $E_\infty^{3,2} \cong \mathbb{Z}_2$
- This is the same as saying that $H^5(F) \cong \mathbb{Z}_2$
- Now we can "drop torsion a degree" and obtain:

$$H_4(F) = \pi_4(F) = \pi_4(S^3) \cong \mathbb{Z}_2$$

What else?

These computations may seem long, but hopefully it shows the extent of the applications of spectral sequences. We have barely scratched the surface in this talk, however, and other applications of the Serre spectral sequence are:

- The Wang sequence
- The Gysin sequence
- Many other computations of cohomology groups
- Homotopy groups of spheres

Exercises

1. Given that $U(1) \cong S^1$ and we have a fibration $U(n-1) \hookrightarrow U(n) \to S^{2n-1}$, compute $H^*(U(n))$
2. Compute $H^*(\Omega S^n)$ using the fibration $\Omega X \to PX \simeq * \to X$ for $X = S^n$
3. Compute the cohomology of the infinite lens space $L(n,q) = S^{2n-1}/\mathbb{Z}_q$, using the fibration $S^1 \to L(n,q) \to \mathbb{CP}^n$

Spectra

Before we move onto the Atiyah Hirezbruch spectral sequence, we look at generalised cohomology and spectra. There are many reasons for studying spectra:

- Homotopy groups of spectra often represent naturally occurring invariants in topology, like algebraic K theory.
- From the commutative algebra perspective, we note that many of the representing spectra carry extra structure that can make them a ring, in a "suitable category of spectra". If we equip this category with something resembling the tensor product, we end up with something that looks like a derived category. These ring spectra are of much interest, but not for this talk.
- Spectra represent generalised cohomology. Indeed, Brown representability asserts that any homotopy functor E^* that satisfies the Eilenburg MacLane axioms can be written as: $E^* = [X, E]_*$, where E is a spectrum.

Defining Spectra

Definition

A spectrum (E_n, ϵ_n) is a sequence $\{E_n\}_{n\in\mathbb{Z}}$ along with maps

$$\epsilon_n : \Sigma E_n \to E_{n+1}.$$

- Since Σ and Ω are adjoint, a map $\epsilon_n : \Sigma E_n \to E_{n+1}$ is equivalent to giving a map $\tilde{\epsilon}_n : \Omega E_n \to E_{n+1}$.
- A Ω-spectrum is a spectrum where $\tilde{\epsilon}_n : \Omega E_n \to E_{n+1}$ is a homotopy equivalence

Examples: (Can you think of any?)

Generalised Cohomology Theories (Examples)

- For an abelian group G, we write HG for the Eilenburg Maclane spectrum, which represents singular cohomology

$$H^n(X;G) = [X, K(G,n)]_*$$

- Complex K theory: Let $KU_{2n} = \mathbb{Z} \times BU$ and $KU_{2n+1} = \Omega BU$. Then:

$$\widetilde{KU}^0 = \tilde{K}^0(X) = [X, KU_0]$$

  - This spectrum is periodic, because Bott periodicity says that $BU \times \mathbb{Z} \simeq \Omega^2 BU$.
- ...Can you think of more?

A Little Bit of History

- One can think of the AHSS as a generalisation of the Serre spectral sequence, to generalised cohomology theories.
- Adams credits the discovery of AHSS to Whitehead, but he is very modest and it was used in a paper of Atiyah and Hirzebruch for the K theory case.

What is it?

Theorem

Given a generalised cohomology theory E^* and a fibration $F \hookrightarrow X \to B$, with B path connected and a CW cell complex. Then there is a spectral sequence called the Atiyah-Hirzebruch spectral sequence with:

$$E_2^{p,q} = H^p(B; E^q(F)) \implies E^*(X)$$

- Note that when $F = *$, we get a fibration $* \to X \to X$, hence a spectral sequence

$$E_2^{p,q} = H^p(X; E^q(*)) \implies E^*(X).$$

K-theory of $\mathbb{CP}^n$

Theorem

$$K^p(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^{p+1}, & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

The E_2 page

We start by determining the E_2 page of our spectral sequence:

$$E_2^{p,q} = H^p(\mathbb{CP}^n; K^q(*)) \implies K^*(\mathbb{CP}^n).$$

It's well known that: $K^q(*) = \begin{cases} \mathbb{Z}, & \text{if } q \text{ is even} \\ 0, & \text{otherwise} \end{cases}$ So our E_2 page becomes:

$$E_2^{p,q} = \begin{cases} H^p(\mathbb{CP}^n), & \text{if } q \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

But this becomes

$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & \text{if } q \text{ and } p \text{ are even, } 0 \leq p \leq 2n \\ 0, & \text{otherwise} \end{cases}$$

Visualising the E_2 page

Recovering the data

Becuase the differentials are of bidegree $(r, 1-r)$, one of these is odd, so all differentials will be trivial. Hence $E_2 = E_\infty$. Hence we can recover $K^m(\mathbb{CP}^n)$:

$$K^m(\mathbb{CP}^n) = \bigoplus_{p+q=n} E_\infty^{p,q} = \begin{cases} \mathbb{Z}^{n+1}, & \text{if } m \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

k-invariants

- Suppose that there is a spectrum with one nontrivial homotopy group $\pi_n(E)$. Then $E \simeq \Sigma^n H\pi_n(E)$ (it's a shift of an Eilenburg Maclane spectrum).
- This works out nicely, but when it has two nontrivial homotopy groups, it doesn't work out so nicely- it need not be a wedge of two shifts of the Eilenburg Maclane spectrum.
- However, not all hope is lost- they fit nicely into a fiber sequence with two Eilenburg Maclane spectra.

How do they fit together?

$$
\begin{array}{ccc}
\Sigma^n H\pi_n(E) & \longrightarrow & E \\
 & & \big\downarrow \varphi \\
 & & H\pi_0(E)
\end{array}
$$

$$
\begin{array}{ccccc}
\Sigma^n H\pi_n(E) & \longrightarrow & E & & \\
 & & \big\downarrow \varphi & & \\
 & & H\pi_0(E) & \xrightarrow{k} & \Sigma^{n+1} H\pi_n(E)
\end{array}
$$

What does this tell us?

- Firstly it helps us to answer the question of when a spectrum E with two nontrivial homotopy groups is a wedge sum of shifts of Eilenburg Maclane spectra; it happens iff $k = 0$.
- For a spectrum E such that $\pi_k(E) = 0$ for $i < k < j$, then there are k-invariants between i and j, by iterating this proceedure.
- k-invariants are examples of stable cohomology operations

Stable cohomology operations

Definition

A stable cohomology operation is a natural transformation $H^n(-;A) \to H^n(-;B)$ which commutes with the suspension.

Examples

- The Bockstein homomorphism β- the connecting homomorphism in the les in homology associated to the ses $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p$.
- Over $\mathbb{Z}$, stable cohomology operations aren't all that interesting.
- However, over $\mathbb{F}_p$, they are very interesting!

Steenrod algebra

- In general, $H^n(X;\mathbb{Z}_p) \to H^{2n}(X;\mathbb{Z}_p)$, $x \mapsto x \smile x$ isn't a homomorphism. However, for $p = 2$, it is! Yet it still isn't natural, or stable. So we amend this with the Steenrod squares.
- The set of cohomology operations $H^*(X;\mathbb{Z}_2) \to H^{*+n}(X;\mathbb{Z}_2)$ form a graded $\mathbb{Z}_2$ algebra under composition generated by the Steenrod squares

$$\mathrm{Sq}^n : H^*(X;\mathbb{Z}_2) \to H^{*+n}(X;\mathbb{Z}_2).$$

Definition of the Steenrod squares

Definition (Steenrod squares)

1. They are group homomorphisms, natural and stable
2. $\mathrm{Sq}^0 = \mathrm{id}$ and $\mathrm{Sq}^1 = \beta$, the Bockstein associated to

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

3. When $|x| = n$, $\mathrm{Sq}^n(x) = x \smile x$
4. When $|x| < n$, $\mathrm{Sq}^n(x) = 0$
5. $\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y)$

Quick facts

Theorem

These relations uniquely define the Steenrod squares and their action on mod 2 cohomology of spaces.

Theorem (Adem relation)

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k$$

Back to AHSS...

If we have a spectrum E with $\pi_q(E)$ and $\pi_{q+r}(E)$ nontrivial but $\pi_k(E)$ trivial for all $q < k < q + r$ then:

Theorem

The first nontrivial differential in the cohomological AHSS from $E_{r+1}^{p,-q} \to E_{r+1}^{p+r,-r-q}$ is identified with the k-invariant

$$H^p(-;\pi_q(E)) \to H^{p+r+1}(-;\pi_{q+r}(E)).$$

- This often enough for us
- For higher differentials, they are determined by higher cohomology operations.

K-theory

- Complex K theory admits one k-invariant, since it is 2-periodic.
- The k-invariant is given by $\beta \circ \mathrm{Sq}^2 \circ r : H^*(-;\mathbb{Z}) \to H^{*+3}(-;\mathbb{Z})$
- Due to the nature of the zeros of real K-theory's homotopy groups and its 8 periodicity, we get 4 k-invariants:
  - $\mathrm{Sq}^2 \circ r$
  - Sq^2
  - $\beta \circ \mathrm{Sq}^2$
  - $\beta \circ \mathrm{Sq}^4$

Stable Homotopy Theory over $\mathbb{Q}$

- It turns out that stable homotopy operations are trivial over $\mathbb{Q}$
- This in turn means that AHSS is much simpler over $\mathbb{Q}$
- In fact, it is so much nicer that all extension problems and differentials are trivial!
- Even more strongly, the ∞ category of rational spectra is equivalent to the ∞-category of chain complexes over $\mathbb{Q}$.

Further directions

Again, we have barely touched the surface:

1. I didn't talk at all about a very large area where this is applicable- bordism.
   - For unoriented bordism, Thom showed that MO is a wedge sum of shifts of the Eilenburg Maclane spectrum. Therefore the k-invariants are trivial and the AHSS collapses at the E_2 page without having to think about extension problems.
2. Homotopy groups of spheres
   - As with most spectral sequences, one can apply them to obtain results about homotopy groups of spheres. The AHSS is not an exception.