Introduction to ∞-Categories

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Notational Remark

Following the notation used by Lurie, I shall be denoting by the models of $(\infty, 1)$-categories in these slides by ∞-categories.

The Plan...

1. Introduction
2. Basics of ∞-categories
   - Some Basics of Simplicial Sets
   - The Definition of an ∞-category
   - Making the Plan Precise
3. What Next?
   - Constructions on ∞-categories
   - Higher Algebra
   - Goodwillie Calculus

The Fundamental groupoid

Before I begin the talk, I would like to start off by giving a rough sketch of the fundamental ∞-groupoid.

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Recall that the fundamental groupoid of a space X $\pi_{\leq 1}(X)$ is a groupoid, whose objects are points in X and whose morphisms are homotopy classes of paths $x \to y$ relative to basepoints.

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- This is a groupoid, since all paths admit an inverse up to homotopy.
- However, it discards a lot of information. A better version is the fundamental ∞-groupoid, $\pi_{<\infty}(X)$

Description of the Fundamental ∞-groupoid

Roughly, we construct the fundamental ∞-groupoid of a space X, $\pi_{<\infty}(X)$ as follows:

- Objects are given by points in X
- Morphisms are paths $x \to y$
- 2-morphisms are given by homotopies between these paths
- Higher morphisms are given by higher homotopies

Why Do We Call it a Groupoid?

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  - Morphisms are paths $x \to y$
  - 2-morphisms are given by homotopies between these paths
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$\Big\}$ Invertible in a weak sense

Introduction

- Category theory is a very useful language, but it certain limitations.
- In mathematics, we often would like to identify two objects which are not isomorphic, but "weakly equivalent" in the homotopy theoretic sense.
- For example, in homological algebra it is desirable to consider chain complexes up to quasi isomorphism. Furthermore in homotopy theory, we also would like to consider weakly homotopic spaces as isomorphic.

Weak Equivalences

Convenient languages for a pair $(\mathcal{C}, \mathcal{W})$ of a category along with a class of morphisms called weak equivalences which we also want to be thought of as isomorphisms have been searched for. These include:

- Model categories
- Derivators
- Simplicial Categories
- Topological categories
- ∞-categories

Weak Equivalences

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- Derivators
  - Simplicial Categories
  - Topological categories
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$\left.\right\}$ All part the theory of $(\infty, 1)$-categories

The Information We Want

We would like certain information in an $(\infty, 1)$-category:

- A class of objects
- Morphisms between objects, 2-morphisms between morphisms, 3-morphisms between 2-morphisms and so forth. This explains why there is an ∞ in the name
- Morphisms can be composed in an associative and unital way.
- Higher morphisms (2-morphisms, 3-morphisms and so forth) should be invertible, at least up to higher morphisms, corresponding to the 1 part of the name.

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} hard to make precise

The Idea...

Recall the following principle in higher category theory known as the homotopy hypothesis:

Spaces and ∞-groupoids should be the same

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Use "simplicial models for spaces", called Kan complexes.

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- We will do so by defining ∞-categories as simplicial sets satisfying certain "horn extension properties"

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- With this approach to ∞-categories we get a model for the fundamental ∞-groupoid

It turns out to be the singular complex $Sing(X)$

The Plan...

Some Basics of Simplicial Sets

Definition

Recall that Δ is the simplicial category:

- Objects are given by totally ordered sets $[n] = \{1, 2, \dots n\}, n \in \mathbb{N}$
- Order preserving functions $[n] \to [m]$ as morphisms

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- Dually, we denote by $c\mathcal{S}et$ the category of cosimplicial sets, given by $\mathbf{Fun}(\Delta, \mathcal{S}et)$

Face and Degeneracy Maps

Definition

We may define the maps:

- $d^i : [n-1] \to [n] \; 0 \leq i \leq n$ (called the face maps)
- $s^j : [n] \to [n-1] \; 0 \leq j \leq n$ (called the degeneracy maps)

as follows:

- d^i is the unique map injective map which skips i in its image
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- Exercise: come up with an explicit description of d^i and s^j using piecewise functions.

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The Data of Simplicial Sets

We may use face and degeneracy maps to rewrite the definition of a simplicial set:

Theorem

A simplicial set X_* is a collection of sets X_n along with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ such that they satisfy:

$$\left.\begin{array}{l}
\bullet\ d_i d_j = d_{j-1} d_i \quad \text{if } i < j \\
\bullet\ d_i s_j = s_{j-1} d_i \quad \text{if } i < j \\
\bullet\ d_i s_j = s_j d_{i-1} \quad \text{if } i > j+1 \\
\bullet\ s_i s_j = s_{j+1} s_i \quad \text{if } i \leq j \\
\bullet\ d_j s_j = d_{j+1} s_j = 1
\end{array}\right\} \textit{simplicial identities}$$

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- More precisely, the kth n-horn is given by the coequaliser:

$$\coprod_{0\leq i<j\leq n} \Delta^{n-2} \rightrightarrows \coprod_{i\neq k} \Delta^{n-1} \to \Lambda^n_k$$

Examples of Horns for $n = 2$

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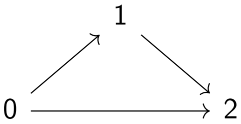

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- $$\Delta^2 = \begin{array}{ccc} & 1 & \\ \nearrow & & \searrow \\ 0 & \longrightarrow & 2 \end{array}$$
- $$\Lambda^2_0 = \begin{array}{ccc} & 1 & \\ \nearrow & & \\ 0 & \longrightarrow & 2 \end{array} \subset \Delta^2$$

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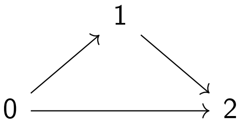

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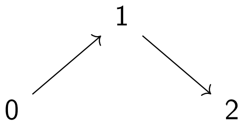

- Exercise: similarly represent Λ^2_k for $k = 1, 2$

The Nerve Functor

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- The nerve functor $N : \mathcal{C}at \to s\mathcal{S}et$ is fully faithful, hence includes an equivalence of categories on its essential image.
- In order to study the essential image, we will exploit certain "horn extension properties" of the nerve functor.

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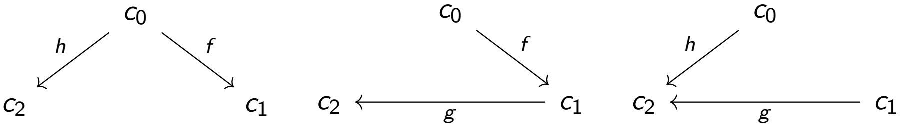
- Writing c_m for the image of m under a horn $\alpha : \Lambda^n_k \to N(\mathcal{C})$ makes the horns $\alpha : \Lambda^2_k \to N(\mathcal{C})$ for $0 \leq k \leq 2$ respectively look like:



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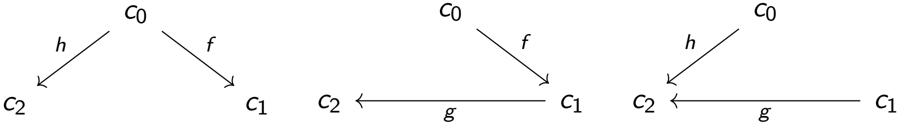
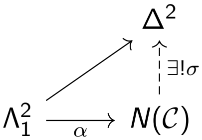


- Using the composition $h = g \circ f$, we see that any horn $\Lambda^2_1 \to N(\mathcal{C})$ extends uniquely to a 2-simplex $\sigma : \Delta^2 \to N(\mathcal{C})$.



Inner and Outer Horns

- Quick remark: the horns Λ_k^n $0 < k < n$ are called inner horns, whereas Λ_0^n and Λ_n^n are called outer horns.

Theorem

Let X be a simplicial set.

- *We have an isomorphism $X \cong N(\mathcal{C})$ $\mathcal{C} \in \mathcal{C}at$ iff every inner horn $\Lambda_k^n \to N(\mathcal{C})$ can be uniquely extended to an n-simplex $\Delta^n \to X$*
- *We have an isomorphism $\mathcal{G} \cong N(\mathcal{G})$ $\mathcal{G} \in \mathcal{G}rpd$ iff every horn $\Lambda_k^n \to N(\mathcal{C})$ can be uniquely extended to an n-simplex $\Delta^n \to X$*

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- The characterisation of the essential image of the nerve functor $N : \mathcal{G}rpd \to s\mathcal{S}et$ inspires the definition of a Kan complex:

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- Denoting by $\mathcal{K}an \subset s\mathcal{S}et$ the full subcategory spanned by Kan complexes, we have a diagram of fully faithful functors:

$$\begin{array}{ccc} \mathcal{G}rpd & \longrightarrow & \mathcal{C}at \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ \mathcal{K}an & \longrightarrow & s\mathcal{S}et \end{array}$$

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- First, for Kan complexes, all horns can be extended, whereas for nerves this is only the case for inner horns.

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- Both Kan complexes and nerves of categories satisfy horn extension properties, but there are two big differences
- First, for Kan complexes, all horns can be extended, whereas for nerves this is only the case for inner horns.
- For Kan complexes, the mere existence of such and extension is required, whereas for nerves this extension must be unique. For our definition of ∞-categories, mere existence is all that is necessary, since we would like all the choices of compositions that exist to be "homotopically irrelavent", similar to concatinating paths in a space X.

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- Any Kan complex will be an ∞-category. Therefore for any space X, it's singular complex $Sing(X)$ is an ∞-category. This is one description of the fundamental ∞-groupoid as said before.
- One thing one can say is that all categories should be ∞-categories too, since all higher morphisms can be given by the identity. While this is in some sense "cheating", it is not wrong. The nerve makes this idea precise- identifying a category $\mathcal{C}$ to its nerve $N(\mathcal{C})$, allows us to studying categories as a special case of higher category theory.

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  - As is normal category theory, we write $f : x \to y$ if $s(f) = x$ and $t(f) = y$. To be extremely precise, we write $hom_{\mathcal{C}}(x, y)$ as the pullback

$$\begin{array}{ccc} hom_{\mathcal{C}}(x,y) & \longrightarrow & \mathcal{C}_1 \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ * & \xrightarrow{(x,y)} & \mathcal{C}_0 \times \mathcal{C}_0 \end{array}$$

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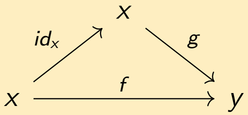
- We write $s_0 : \mathcal{C}_0 \to \mathcal{C}_1$, $x \mapsto \mathrm{id}_x = s_0(x)$ for the identity map

Homotopies in ∞-categories

- We write $\partial\sigma = (d_0\sigma, \ldots, d_n\sigma)$ for the boundary of an n-simplex

Definition

We say that two morphisms $f, g : x \to y$ are homotopic if there is a two simplex $\sigma : \Delta^2 \to \mathcal{C}$ with boundary $\partial\sigma = (g, f, id_x)$; so the boundary looks like:



Homotopy is an Equivalence Relation

Theorem

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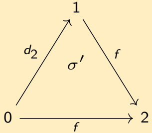
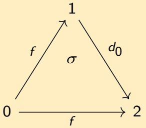
Proof.

First we show that $f \simeq f$. Consider an edge $f : \Delta^1 \to \mathcal{C}$. Then there are 2-simplices σ, σ' such that $d^0(\sigma)$ and $d^2(\sigma')$ are degenerate.





Note that $d^1(\sigma) = d^2(\sigma) = f$ and $d^0(\sigma') = d^1(\sigma') = f$ and so

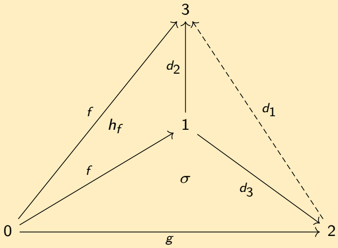
$$s^0(f) = \sigma, \quad s^1(f) = \sigma' \implies f \simeq f$$

□

Proof of Symmetry

Proof.

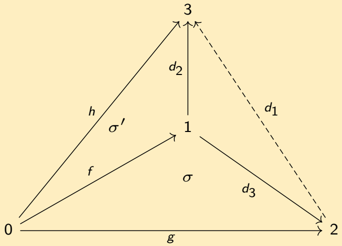
Let $\sigma : f \simeq g$ be a homotopy and by the previous part of the proof there is a homotopy $h_f : f \simeq f$, so we have:



One can see that $\tilde{\sigma} = d_1 : g \simeq f$ is our desired homotopy. □

Proof.

Observe that symmetry in this case is just a special case of the proof of transitivity; replacing $h_f : f \simeq f$ with $\sigma : f \simeq g$ and $\sigma : f \simeq g$ with $\sigma' : g \simeq h$, we get:



□

Compositions of Morphisms

Definition

Let $f : x \to y$ and $g : y \to z$ be morphisms in an arbitrary ∞-category $\mathcal{C}$. Then, to compose these two morphisms to form an inner horn in $\mathcal{C}$

$$\lambda = (g, \bullet, f) : \Lambda_1^2 \to \mathcal{C}$$

such that $d_0\lambda = g$ and $d_2\lambda = f$. Any such horn can be non-uniquely extended to a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$. The new face of σ, $d_1(\sigma)$ is called a candidate composition of g and f.

Compositions of Morphisms

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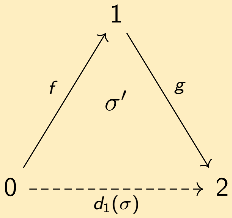

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- Clearly this is very different from normal category theory, because we don't require our candidate composition to be unique.
- To re-emphasise, we only require in higher category theory that all of these candidate compositions are weakly equivalent in the homotopy theoretic sense.
- We shall now form the homotopy category $\mathrm{Ho}(\mathcal{C})$ of an ∞-category, and show that it is well defined.

The Homotopy Category of an ∞-category

- Now that we have a well defined homotopy relation, we may form the homotopy category $\mathrm{Ho}(\mathcal{C})$ of an ∞-category by passing everything to homotopy classes.

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- Now that we have a well defined homotopy relation, we may form the homotopy category $\mathrm{Ho}(\mathcal{C})$ of an ∞-category by passing everything to homotopy classes.
- Various things must be verified in order to show that $\mathrm{Ho}(\mathcal{C})$ is well defined, but the main one is to show that every candidate composition of two morphisms are indeed homotopic.

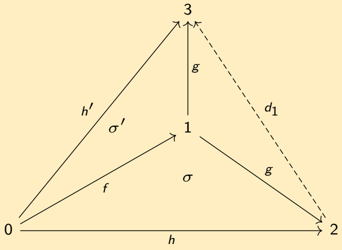
The Homotopy Category is Well Defined

Theorem

Given two morphisms $f : x \to y$ *and* $g : y \to z$ *with two 2-simplices* $\sigma, \sigma' : \Delta^2 \to \mathcal{C}$ *along with* $h = d_1(\sigma)$ *and* $h' = d_1(\sigma')$. *Seeing that* h, h' *are both candidate compositions of* f *and* g, $h \simeq h'$.

Proof.

Consider the diagram, where the extension property of ∞-categories guarantees that we may extend the horn $\lambda : \Lambda_1^3 \to \mathcal{C}$ which is pictured with the solid arrows to a 3-simplex $\tau : \Delta^3 \to \mathcal{C}$ pictured:



Clearly, $d_1(\tau) : h \simeq h'$ is a homotopy. □

The Definition of the Homotopy Category

Definition

Let $\mathcal{C}$ be an ∞-category. Then there is an ordinary category $\mathrm{Ho}(\mathcal{C})$, called the homotopy category of $\mathcal{C}$, with:

- $\mathrm{Ob}(\mathrm{Ho}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$
- Morphisms are given by homotopy classes of maps in $\mathcal{C}$. The composition and identites respectively are:
  - $[g] \circ [f] = [g \circ f]$, where $g \circ f$ is an arbitrary candidate composition.
  - $\mathrm{id}_x = [\mathrm{id}_x] = [s_0(x)]$

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- Our first principle which we want is that:
  - A composition of f and g exists
  - Any two choices of compositions are homotopic; the space of such compositions has trivial π_0.
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Frame Title

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    - And so forth (trivial $\pi_i,\ i > 2$)
- We want morphisms of higher dimensions. If $\{i_0, \dots, i_k\} \in \Delta^n$ are vertices, then we may denote by $\Delta^{\{i_0, \dots i_k\}}$ the k-simplex of Δ^n spanned by the vertices $\{i_0, \dots, i_k\}$ then:
    - An n-morphism from $x \to y$ is given by a map of simplicial sets $\tau : \Delta^{n+1} \to \mathcal{C}$ such that $\tau|_{\Delta^{\{0,\dots,n\}}} = x$ and $\tau|_{\Delta^{\{n+1\}}} = y$

Making The Plan Precise

To convince ourselves that we have done the first part, consider the following theorem of Joyal:

Theorem

A simplicial set X is an ∞-category iff the restriction map

$$i^* : \mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$$

is an acylic Kan fibration.

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- We may think of $\mathrm{Map}(\Lambda_1^2, X)$ as the space of all composition problems, whereas $\mathrm{Map}(\Delta^2, X)$ is the space of all solutions to the composition problems.
- What the theorem tells us is that the defining feature of an ∞-category is that these two spaces are homotopically equivalent.
- Moreover, if we have two morphisms $f : x \to y$ and $g : y \to z$ which are composible in an ∞-category $\mathcal{C}$, then we can form the associated horn

  $$\lambda = (g, -, f) : \Lambda_1^2 \to \mathcal{C}$$

  which is just a vertex $\Delta^0 \to \mathrm{Map}(\Lambda_1^2, \mathcal{C})$. Then the fiber F_λ of i^* over the vertex, which may be thought of as the space of all possible compositions of g and f which, by the theorem is a contractible Kan complex, which tells us the information we wanted in the first condition.

Equivalences in an ∞-category

Now, we may just say composition instead of "candidate composition". We can now deal with equivalences in an ∞-category $\mathcal{C}$:

Definition

A morphism $f : x \to y$ in $\mathcal{C}$ is an equivalence if $[f] : x \to y$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$

Back to Groupoids and the Homotopy Hypothesis...

Recall the homotopy hypothesis in the introduction which says that ∞-groupoids should come from spaces. First, we need a definition of an ∞-groupoid:

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An ∞-category is an ∞-groupoid if the homotopy category is a groupoid.

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- The following result of Joyal says that all horns can be extended for an ∞-category as soon as certain maps are equivalences:

Theorem

Let $\mathcal{C}$ be an ∞-category. Any horn $\lambda : \Lambda_0^n \to \mathcal{C}$, $n \geq 2$ such that $\lambda|_{\Delta^{\{0,1\}}}$ can be extended to a simplex $\Delta^n \to \mathcal{C}$. A similar statement can be given for Λ_n^n

Some Closing Remarks:

We have a corollary now which corresponds perfectly with the homotopy hypothesis:

Corollary

An ∞-category is an ∞-groupoid iff it is a Kan complex

Therefore, we can refine our original diagram of faithful functors to be:

$$\begin{array}{ccccc} \mathcal{G}rpd & \longrightarrow & \mathcal{C}at & & \\ \downarrow N & & \downarrow N & \searrow^{N} & \\ \mathcal{K}an = \mathcal{G}rpd_\infty & \longrightarrow & \mathcal{C}at_\infty & \longrightarrow & s\mathcal{S}et \end{array}$$

The Plan...

- In the next slides, I shall give a very rough overview of the types of things which one can do, having a precise model of the $(\infty, 1)$-category.

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- These have been studied ,of course, and now we may speak comfortably about (co)limits in ∞-categories. These are an important topic in 1-category theory, so it is natural to study them in an ∞-categorical setting too.

Overview of Higher Algebra

- One of the main objects in higher algebra are called $\mathbb{E}_\infty$-rings. Roughly, what they are are a space X which satisfy the ring axioms up to coherent homotopy.

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- $\mathbb{E}_\infty$-rings can be thought of as playing the same role in stable homotopy theory as commutative rings do in algebra.
- The collection of all spectra can be arranged into an ∞-category, which can be thought of as the ∞-categorical version of abelian groups. The tensor product on abelian groups has the analogue of the smash product.
- One may, for example, introduce the notion of a stable ∞-category, which is essentially an axiomisation of the essential principle in stable homotopy theory which is that fiber sequences and cofiber sequences are the same. Furthermore, the ∞-category of spectra are an example of an ∞-category.

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- The theory starts with the categorification of polynomial functions, called n-excisive functors, $P_n F$. Goodwillie established that every homotopy functor F has a universal approximation by these functors.
- For example, it turns out that 1-excisive functors represent generalised homology theories (roughly). Consider $F = I$, the identity functor on the category of based spaces. In the Goodwillie calculus, this functor is highly nontrivial; $P_1 I(X) = \Omega^\infty \Sigma^\infty X$. It represents stable homotopy theory in the sense that $\pi_*(P_1 I(X)) \cong \pi_*^s(X)$. As you climb higher up the tower $P_n I(X)$, it will interlope between stable and unstable homotopy theory, satisfying various higher versions of the excision axiom.

Thank you for watching!