

K Theory 1- Various Interpretations

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Introduction

I will be beginning where I left off in my previous fiber bundles and homotopy theory talk. At the end of the talk, I defined the Grothendieck group of a monoid. Noticing that $Vect(X)$ is a monoid with \oplus , Whitney sum, and defined the K theory of a space X by $K(X) = K(Vect(X))$. In this talk, I shall give some further definitions, and methods for computing K -theory groups. We shall use the reduced K theory groups and some related sequences to show the following important isomorphisms: $K_{\mathbb{R}}(X) \cong [X, \mathbb{Z} \times BO]$ and $K_{\mathbb{C}}(X) \cong [X, \mathbb{Z} \times BU]$. Then, I shall look at K theory as a cohomology theory (proving that it is in fact a generalised cohomology theory) and some cohomological properties of K theory, stating Bott periodicity both homotopically and cohomologically. All spaces are assumed to be compact.

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The Reduced K -theory, $\tilde{K}(X)$

Since the functor K is contravariant in the category of compact spaces, the inclusion from $p \rightarrow X$ induces a map $\alpha : K(X) \rightarrow K(p) \cong \mathbb{Z}$

Definition

The kernel, $\ker[\alpha : K(X) \rightarrow \mathbb{Z}]$ of α is denoted by $\tilde{K}(X)$ and is called the reduced K theory of X .

The Reduced K theory

Theorem

If $\beta : X \rightarrow p$ is the collapsing then the induced map β^ induces a natural splitting*

$$K(X) \cong \tilde{K}(X) \oplus K(p) \cong \tilde{K}(X) \oplus \mathbb{Z} \quad (1)$$

A relation to Čech cohomology

For every vector bundle E , note that there is a locally constant function $r : X \rightarrow \mathbb{N}, x \rightarrow \dim(E_x)$. Now let $H^0(X; \mathbb{N})$ be the monoid of all such locally constant functions. There is no reason that we can't take the Grothendieck group of both $Vect(X)$ and $H^0(X; \mathbb{N})$ and look at the induced map which r defines from $Vect(X) \rightarrow H^0(X; \mathbb{N})$. The Grothendieck group of $H^0(X; \mathbb{N})$ becomes the first Čech cohomology group of X . $H^0(X; \mathbb{Z})$ - the locally constant functions with values in \mathbb{Z} . So we have a group homomorphism

$$r : K(X) \rightarrow H^0(X; \mathbb{Z}) \quad (2)$$

Definition

We let $K'(X) = \ker[r : K(X) \rightarrow H^0(X; \mathbb{Z})]$

It turns out that $K'(X)$ is closely related to $\tilde{K}(X)$:

$K'(X)$ vs $\tilde{K}(X)$

Theorem

If X is compact we have a split exact sequence

$$0 \rightarrow K'(X) \rightarrow K(X) \xrightarrow{r} H^0(X; \mathbb{Z}) \rightarrow 0 \quad (3)$$

which admits a canonical splitting. Furthermore, if X is connected, we have a canonical isomorphism $K'(X) \cong \tilde{K}(X)$

Proof.

Omitted. □

A Preliminary Theorem

The goal of the next few slides is to show an isomorphism $\varinjlim Vect_n(X) \cong K'(X)$. In order to do so, we will need the following preliminary theorem:

Theorem

Let θ_n be the trivial bundle of rank n over X . Then every $x \in K(X)$ is of the form

$$[E] - [\theta_n] \tag{4}$$

for some n . Furthermore, we have that $[E] - [\theta_n] = [F] - [\theta_m]$ iff there is a $p \in \mathbb{Z}$ such that

$$E \oplus \theta_{m+p} \cong F \oplus \theta_{n+p} \tag{5}$$

Studying $\varinjlim Vect_n(X)$

The point of this slide is to show an isomorphism $\varinjlim Vect_n(X) \cong K'(X)$, where the $_n$ just means that we are classifying the isomorphism classes of complex vector bundles of rank n over X . Now, we can put an abelian monoid structure on $\varinjlim Vect_n(X)$ via the Whitney sum:

$$Vect_n(X) \times Vect_p(X) \rightarrow Vect_{n+p}(X) \quad (6)$$

If we write \dot{E} for the class of E in $Vect_n(X)$, we may see that $[E] - [\theta_n] \in \ker(r : K(X) \rightarrow H^0(X; \mathbb{Z}))$. This correspondence $\dot{E} \mapsto [E] - [\theta_n]$ induces a monoid homomorphism

$$\varphi : \varinjlim Vect_n(X) \rightarrow K'(X) \quad (7)$$

which we will see is in fact an isomorphism.

φ is an isomorphism

Theorem

φ is an isomorphism. Hence $\varinjlim Vect_n$ is a group.

Proof

Proof.

First let's show that it's injective. By our preliminary theorem, we have that $[E] - [\theta_n] = [F] - [\theta_m]$ iff

$$E \oplus \theta_{m+p} \cong F \oplus \theta_{n+p} \quad (8)$$

and so the map $\varinjlim Vect_n(X) \rightarrow K'(X)$ is injective. Now we must show surjectivity. Let $x \in K'(X)$. Yet again, we can use the preliminary lemma to say that x is of the form $[E] - [\theta_n]$, where

$$r([E] - [\theta_n]) = 0 \quad (9)$$

that is, $\dim(E_x) = n$ for all x . Hence φ is surjective as required. \square

Some Ground Work for an Important Isomorphism for $K'(X)$

Before we can look at the homotopical view of K -theory, we must first go over a few results. Our first one is this:

Theorem

We have the following isomorphisms:

$$\text{Vect}_n^{\mathbb{R}}(X) \cong [X, BO(n)] \quad (10)$$

$$\text{Vect}_n^{\mathbb{C}}(X) \cong [X, BU(n)] \quad (11)$$

for every compact space X .

Proof.

Omitted. □

The Homotopical Point of View (for $K'(X)$)

Theorem

Let $BO = \varinjlim BO(n)$ and $BU = \varinjlim BU(n)$. Then we have a natural isomorphisms:

$$K'_{\mathbb{R}}(X) \cong [X, BO] \quad (12)$$

$$K'_{\mathbb{C}}(X) \cong [X, BU] \quad (13)$$

Proof

Proof.

Note that $\varinjlim [X, BO(n)] \cong [X, BO]$. Furthermore, recall that we have

$$Vect_n^{\mathbb{R}}(X) \cong [X, BO(n)] \quad (14)$$

and note that the map $Vect_n^{\mathbb{R}}(X) \rightarrow Vect_{n+1}^{\mathbb{R}}(X)$ induced by adding the trivial bundle of rank one coincides with the inclusion $BO(n) \hookrightarrow BO(n+1)$, so overall we have the following:

$$K'_{\mathbb{R}}(X) \cong \varinjlim Vect_n^{\mathbb{R}}(X) \cong \varinjlim [X, BO(n)] \cong [X, BO] \quad (15)$$

as required. The complex case is analagous. □

The Homotopical View of K Theory

Theorem

For compact spaces X , we have isomorphisms

$$K_{\mathbb{R}}(X) \cong [X, \mathbb{Z} \times BO] \quad (16)$$

and

$$K_{\mathbb{C}}(X) \cong [X, \mathbb{Z} \times BU] \quad (17)$$

Proof

Proof.

Recall that we have a splitting

$K(X) \cong K'(X) \oplus H^0(X; \mathbb{Z}) \cong K'(X) \oplus [X, \mathbb{Z}]$ For complex K -theory this splitting becomes

$$K_{\mathbb{C}}(X) \cong [X, BU] \times [X, \mathbb{Z}] \cong [X, BU \times \mathbb{Z}] \quad (18)$$

and similarly for real K -theory, we have

$$K_{\mathbb{R}}(X) \cong [X, BO] \times [X, \mathbb{Z}] \cong [X, BO \times \mathbb{Z}] \quad (19)$$

as required. □

Bott Periodicity

I feel that I should wrap up this section by stating Bott periodicity from a more homotopy theoretical point of view:

Theorem (Bott Periodicity)

We have that

$$\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO) \implies O \simeq \Omega^8 O \quad (20)$$

and that

$$\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU) \implies U \simeq \Omega^2 U \quad (21)$$

Bott Periodicity, continued

Furthermore, we have the following periodic homotopy groups:

$$\pi_0(O) = \mathbb{Z}/(2) \quad (22)$$

$$\pi_1(O) = \mathbb{Z}/(2) \quad (23)$$

$$\pi_2(O) = 0 \quad (24)$$

$$\pi_3(O) = \mathbb{Z} \quad (25)$$

$$\pi_4(O) = 0 \quad (26)$$

$$\pi_5(O) = 0 \quad (27)$$

$$\pi_6(O) = 0 \quad (28)$$

$$\pi_7(O) = \mathbb{Z} \quad (29)$$

This, coupled with $\pi_n(O) \cong \pi_{n+8}(O)$ has computed the homotopy groups of the infinite orthogonal group. Similarly, we have $\pi_i(U) = 0$ for i even and $\pi_i(U) = \mathbb{Z}$ for i odd.

Generalised Cohomology Theories

K theory can, with a bit of work, model some cohomological properties. We shall put $K(X) = K^0(X)$ and try to define some more K groups, which have nice properties. It turns out that, after doing this work, K -theory is a generalised cohomology theory.

Definition

We denote by X^+ the space X disjoint union with a point. Note that $K(X) = \tilde{K}(X^+)$. Then define



$$\tilde{K}^{-n}(X) = \tilde{K}(\Sigma^n X), X \in \mathcal{C}^+ \quad (30)$$



$$K^{-n}(X, Y) = \tilde{K}^{-n}(X/Y) = \tilde{K}^{-n}(\Sigma^n(X/Y)), (X, Y) \in \mathcal{C}^2 \quad (31)$$



$$K^{-n}(X) = K^{-n}(X, \emptyset) = \tilde{K}(\Sigma^n(X/\emptyset)) = \tilde{K}(\Sigma^n(X^+)), X \in \mathcal{C} \quad (32)$$

Bott Periodicity Again

Theorem (Bott Periodicity)

We have that:

$$K_{\mathbb{C}}^{-i}(X) \cong K_{\mathbb{C}}^{-i-2}(X) \quad (33)$$

$$K_{\mathbb{R}}^{-i}(X) \cong K_{\mathbb{R}}^{-i-8}(X) \quad (34)$$

Proof.

We give the proof for $i = 0$. Note that $\tilde{K}_{\mathbb{C}}^{-i}(X) \cong [\Sigma^i X, \mathbb{Z} \times BU]'$ (homotopy classes of maps which preserve basepoints). Now we have:

$$\tilde{K}_{\mathbb{C}}^{-2}(X) \cong [\Sigma^2 X, \mathbb{Z} \times BU]' \quad (35)$$

$$\cong [X, \Omega^2(\mathbb{Z} \times BU)]' \quad (36)$$

$$\cong [X, \mathbb{Z} \times BU]' \quad (37)$$

$$\cong \tilde{K}_{\mathbb{C}}(X) \quad (38)$$

The real case is analogous. □

K -Groups of Spheres

Theorem (K -Groups of Spheres)

We have the following isomorphisms:

$$\tilde{K}_{\mathbb{C}}^0(S^{2k}) \cong \mathbb{Z} \quad (39)$$

$$\tilde{K}_{\mathbb{C}}^{-1}(S^{2k}) \cong 0 \quad (40)$$

$$K_{\mathbb{C}}^0(S^{2k}) \cong \mathbb{Z} \oplus \mathbb{Z} \quad (41)$$

$$K_{\mathbb{C}}^{-1}(S^{2k}) \cong \mathbb{Z} \quad (42)$$

Proof.

$$\tilde{K}_{\mathbb{C}}^{-1}(S^{2k}) \cong [S^{2k+1}, \mathbb{Z} \times BU]' = \pi_{2k+1}(BU) \cong \pi_{2(n+1)}(U) \quad (43)$$

which we computed before to be precisely 0. The rest of the proof is analogous. □

Defining $K^n(X, Y)$

Definition

Because of Bott periodicity, we may define:

$$K^n(X, Y) = K^{n-2}(X, Y) \quad (44)$$

We will now finish by seeing which of the Eilenberg Steenrod Axioms hold for these groups.

Checking eilenberg Steenrod I

The correspondence

$$(X, Y) \mapsto K^n(X, Y) \quad (45)$$

is a functor from

$$\mathcal{C}^2 \rightarrow \mathbf{Ab} \quad (46)$$

Checking eilenberg Steenrod II

We do in fact have an exact sequence containing six terms which looks like this:

Checking eilenberg Steenrod III

Clearly we have the excision axiom which states

$$K^n(X, Y) \cong K^n(X/Y, *) \quad (47)$$

since this is how we defined the relative K groups.

Checking eilenberg Steenrod IV

The dimension axiom fails! We have that

$$K^0(*) \cong \mathbb{Z} \not\cong 0, \quad (48)$$

which is a theorem we made use of when defining the reduced K -theory at the start! Note that for odd n , the dimension axiom does hold:

$$K^1(*) \cong 0, \quad (49)$$

Checking eilenberg Steenrod V

We show that homotopic maps $f_0, f_1 : X \rightarrow Y$ induce the same homomorphism $K^0(Y) \rightarrow K^0(X)$ Let $f_0, f_1 : X \rightarrow Y$ be continuous maps which are homotopic. Then f_0 and f_1 induce the same homomorphism

$$Vect(Y) \rightarrow Vect(X) \quad (50)$$

(proof omitted) and hence the same homomorphism

$$K(Y) = K(Vect(Y)) \rightarrow K(X) = K(Vect(X)) \quad (51)$$

To Conclude...

What we have shown in this section is that complex K theory is a generalised cohomology theory. Roughly, this means that it satisfies some of the Eilenberg-Steenrod axioms. Furthermore, it is periodic of order two. Next time, we will look at some methods for computing these groups, and at some point, some applications to other fields of maths!